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ABSTRACT

This study focuses on the mental processes used during estimation. This research: (1) developed an operational definition of estimation; (2) created an instrument to identify exceptionally good estimators; (3) collected performance level data on 1187 subjects; (4) constructed interview problems and developed protocols to identify the thinking processes used by people doing computational estimation; (5) collected interview data from subjects; and (6) synthesized the interview data and organized them into a framework that provides a model for future researchers. Time for response per item was minimized in order to maximize the chance that estimation was measured. An Assessing Computational Estimation test was created based on guidelines gathered from a review of relevant research. Characteristics of good estimators were hypothesized, and both student and adult subjects were selected based on this model. Three key processes, labeled translation, reformulation, and compensation, were identified that seemed closely associated with good estimation skills. Strategies intertwined with these processes included forms of rounding and truncation. Further details of the estimation process and samples of the testing and interview packets are also included. (MP)

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IDENTIFICATION AND CHARACTERIZATION OF COMPUTATIONAL ESTIMATION PROCESSES USED BY INSCHOOL PUPILS AND OUT-OF-SCHOOL ADULTS

Final Report

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The financial support of the National Institute of Education made this research on estimation possible. It is hoped that this final report is viewed as a substantial contribution to research in mathematics education and will stimulate additional research in this important area.

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An examination of this report will show that several thousand people participated in pilot testing or the actual data collection phase. A sincere thanks to the participating schools in the districts of Cape Girardeau and Columbia, Missouri; Escondido, California; and Parkway School District of St. Louis, Missouri. In this regard, Bert Kaufman and Martin Herbert of the Comprehensive School Mathematics Program were most cooperative with their experimental classes in Kirkwood and Webster Groves, Missouri. Their help and thoughtful ideas are appreciated.

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Robert E. Reys
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INTRODUCTION

Rationale

Computational estimation has long been recognized as a basic mathematical skill, and several recent recommendations on basic skills in mathematics have re-emphasized the fundamental importance of estimation. For example, the position paper on basic skills of the National Council of Supervisors of Mathematics (1977) attached central importance to the skills associated with estimation and approximation, as well as alertness to reasonableness of results. Bell's (1974) statement on basic mathematical skills needed by "everyman" made mention of the importance of estimation skills, as did many of the participants in the NIE Conference on Basic Mathematical Skills and Learning (1975) in Euclid, Ohio. The need for developing students' estimation skills was also reflected in An Agenda for Action (National Council of Teachers of Mathematics, 1980). One of the specific recommendations made was that teachers "incorporate estimation activities within all areas of the (school) program on a regular and sustaining basis" (p. 7).

The widespread use and availability of hand calculators places additional importance on computational estimation skills. It is easy to make a keystroking error, such as pressing a wrong key or omitting a decimal point, when entering information in a calculator. A single error can greatly effect the result displayed. Since most hand calculators do not provide printouts, it is difficult to know if faulty data have been entered. Therefore skill in estimating the magnitude of an answer and/or recognizing the reasonableness of results is very important.

Despite the importance of estimation, it is perhaps the most neglected skill area in the mathematics curriculum (Carpenter, Coburn, Reys, and Wilson, 1976). Although computational estimation is traditionally introduced around the fourth grade, it is doubtful that the cursory treatment given to estimation in most mathematics programs is sufficient to build any appreciable estimation skills. Estimation frequently appears as a separate topic that is often poorly motivated or perhaps even ignored in work with computation. A review of mathematics basal textbooks (Skvarcius, 1973) showed that very little attention is given to systematic development of computational estimation skills. Another recent study of three popular mathematics textbook series revealed that estimation appeared in less than three percent of the lessons (Freeman, Kuhs, Belli,Floden, Khappen, Porter, Schmidt, and Schwille, 1980).

The lack of attention to computational estimation was reflected in the low performance of all age groups on estimation exercises in the second mathematics assessment of the National Assessment of Educational Progress (NAEP) (Carpenter, Corbitt, Kepner, Lindquist, and Reys, 1980). Typical of such results are those shown in the following exercise:

> ESTIMATE the answer to 12/13 + 7/8. You will not have time to solve the problem using paper and pencil.

	Percent Correct	
Response Categories	Age 13	Age 17
1	7	8
2*	24	37
19	28	21
21	27	15
I don't know	14	18

*Correct

These results show that only 24 and 37 percent of 13- and 17-year-olds, respectively, responded correctly. Over half of the 13-year-olds and about one-third of the 17-year-olds reported values that were completely unreasonable. Rather than estimate the sum of the two fractions, many students apparently attempted to apply an algorithm to the numbers without checking the reasonableness of their estimate. These levels of performance were consistent with results reported earlier in the National Longitudinal Study of Mathematics Ability (NLSMA) (Wilson, Cahen, and Begle, 1968).

The results of investigations such as NAEP and NLSMA suggest that students' estimation skills are poorly developed and that such skills do not automatically develop from maturation or from the study of more mathematics. The results also highlight the fact that estimation skills are difficult to measure. These problems were documented in a review of research on estimation skills compiled by Buchanan (1978). Buchanan's review indicated a dearth of research into estimation skills, despite the importance of such skills in mathematics. The need to explore computational estimation skills, especially in view of the increasing importance being attached to the development of such skills, provided the rationale for this investigation.

Definition of Terms

The close relationship between computational estimation and mental arithmetic is evidenced by the similarity with which researchers have defined the two terms. Researchers have agreed that both estimation and mental arithmetic are to be accomplished without the use of paper and pencil or other similar tools (Dickey, 1934; Flournoy, 1959 a and b; Good, 1973; Nelson, 1966; Olander and Brown, 1959). Olander and Brown defined mental arithmetic to be "the mental or thought processes pupils engage in when attempting to solve arithmetic examples or problems without the use of paper and pencil". Good, in his 1973 edition of the Dictionary of Education, defined the process of estimation as follows: "to arrive at a value either by inspection without calculating the result or by rough calculation".

The discriminating element between the definitions of estimation and mental arithmetic is that those defining estimation (Dickey, 1934; Nelson, 1966) specify that the desired outcome is an approximation, whereas mental arithmetic is associated with a unique solution or answer to a given problem. Unfortunately, no evidence has been found in the research literature of an attempt to define an approximation or to specify what constitutes an acceptable approximation or estimate. In fact, both Dickey and Nelson used approximation as an undefined term. The Dictionary of Education (1973) defines approximate computation as: "(1) computation that involves the use of approximate numbers; (2) the application of methods or approximation with either approximate or exact numbers".

Since no acceptable or complete definition of computational estimation has been found, the following operational definition was constructed for this investigation:

> Computational Estimation:
>
> The interaction and/or combination of mental computation, number concepts, technical arithmetic skills including rounding, place value, and less straightforward processes such as mental compensation that rapidly and consistently result in answers that are reasonably close to a correctly computed result. This process is done internally, without the external use of a calculating or recording tool.

This definition provided the foundation for the investigations of this project. More specifically, it provided both structure and direction for various procedures that were followed in this research. Perhaps this definition can represent a step toward a common understanding and acceptance of just what constitutes computational estimation.

Objectives of the Study

This report represents the culmination of an intensive year-long study of estimation skills possessed by good estimators at different levels of maturity. More specifically this research:

1. developed an operational definition of computational estimation, and, based on this definition,

2. developed an instrument to identify people who

were exceptionally good in computational estimation;

3. used this instrument to collect performance level data on 1187 subjects;

4. constructed interview problems and developed protocols to identify the thinking processes subjects used when doing computational estimation;

5. collected interview data from subjects;

6. synthesized the interview data and organized them into a framework that provided a model for future researchers to extend.

This research is in the spirit of thoughts expressed by one mathematics educator who wrote that more must be "learned in the next few years about how students develop these (estimation) skills, how this work can best be integrated into the curriculum, and how instruction can more closely fit the psychology of the learner" (Trafton, 1978, p. 213).

LITERATURE REVIEW

Assessment Considerations

In describing estimation results from the first NAEP mathematics assessment, Carpenter, Coburn, Reys, and Wilson (1975), observed that inherent difficulties in assessing computational estimation have hampered evaluation efforts in this area, which in turn negatively affects curricular emphasis and instructional time devoted to developing such skills. In a similar vein, Sachar concluded that "research capabilities are often constrained by measurement and methodological technologies" (1978, p. 237). Thus, the question of how to assess computational estimation skills is crucial. Since some of the same problems are common to the assessment of both mental computation and estimation skills, procedures used to assess one area have implications for the other. Due to the scarcity of previous efforts to assess estimation, it became necessary to examine procedures used to assess mental computation. A review of those studies produced some helpful ideas and needed direction.

Research has been conducted which compares various modes of assessing mental arithmetic ability (Olander and Brown, 1959; Sister Josephina, 1960). These studies present conclusive evidence that some form of written measure computational estimation rather than mental computation or algorithmic skills using paper and pencil. Although it is conceded that mental computation is a component of estimation (Paull, 1971), there is a need for assurance that estimation skills are being assessed. Several studies have indicated, either implicitly or explicitly, that when the time for responses to items is minimized then estimation is more likely to be measured (Brown, 1957; Nelson, 1966; Bestgen et al., 1980). However, tests presented in written form with a fixed time block do not control the time per item, a control which is essential when assessing computational estimation skills. Thus, some researchers (Bestgen et al., 1980; Brown, 1957) have sought innovative modes of time-controlled item presentation.

No evidence exists that there has been an estimation test developed and widely accepted by researchers and psychometricians to measure estimation skills. Thus, most researchers who have attempted to assess estimation skills have constructed their own test instruments. Paull (1971) constructed an estimation test containing 16 items. Each item contained two or more decimal numbers, most with multiple operations. The test was designed for eleventh graders and was limited to 8 minutes administration time. Nelson (1966) instructed the teachers administering her 40-item estimation test to allow a maximum of one hour or the time when 90 percent of the class had finished. She designed the test for fourth and sixth graders and used numbers similar to those used by Brown. In her mental arithmetic test involving subtraction, Brown constructed 26 items, 14 of which used whole numbers and the remaining 12 equally divided among decimals, fractions, denominate numbers, and verbal problems. Her test was designed for grades 6-12 with 5 minutes allowed for administration time. The test used by Bestgen et al. (1980) with pre-service elementary teachers was a 60-item test including only whole and decimal numbers. Five minutes were allowed for test administration, but no subject completed all of the items in the allotted time.

In other studies related to test construction, other features of estimation tests thought to influence results were studied. For example, Buckley (1974) found no beneficial effect of changing the order of addends on a mental arithmetic test. Another researcher (Hall, 1951) discovered that three-step problems were considerably more difficult than two-step problems which, in turn, were more difficult than problems with one step. Olshen (1975) described use of "recovery items," items which have a high probability of being answered correctly, in her mental computation test. Such items were found to restore student confidence and to renew their attention to the task. Finally, although the bulk of computational items on most tests are stated without context, some tests (Brown, 1957; Faulk, 1962) have included problems using numbers in an applied context. For example, Faulk posed the estimation question, "If a gallon of gasoline costs 26¢, how much will 15 gallons cost?"

Identification of Estimation Strategies

The determination of the strategy that an individual uses when deriving an estimate to a problem requires something other than an objective test. Generally, researchers have used either interviews or written explanations (Faulk, 1962) to ascertain the strategy used. Interviews seem to be the most widely accepted mode of determining estimation strategies (Urbatsch, 1979; Corle, 1958; Nelson, 1966; Olander and Brown, 1959). Prior to the interviews, the subjects were instructed to relate verbally their thought processes either while they were giving their response or immediately afterward. Urbatsch (1979) asked her subjects if they could make a closer estimate and if their estimate was higher than, lower than, or equal to the exact answer for each of the 9 problems used in her interview. The probes that Corle (1958) used on 8 problems given to 64 sixth grade pupils in interviews were: (a) What is your answer? (b) How did you get it? (c) Why did you work it that way? (d) Do you think that your answer is right? and (e) Why do you think it is (or is not)? Although Nelson (1966) used various probes in her interviews, she did not report the strategies that subjects verbalized, but instead reported whether the subject was able to estimate correctly and whether wrong answers resulted from wrong processes.

Subject Population

The population from which all reported studies on estimation and mental arithmetic have drawn their samples have been school students from grade 4 through college. Most of these studies have been experimental studies with performance in mental arithmetic as the dependent variable, but findings from these studies have certain implications for the project study of estimation skill. After screening some 1400 students in grades 6-12, Brown (1957) interviewed those who scored in the top 5 percent in each of the seven grades on her mental arithmetic test. She observed that the greatest improvement in mental arithmetic skills took place between the sixth and seventh and seventh and eighth grades. In her study with fourth and sixth graders, Nelson (1966) concluded that the teaching of estimation procedures were more effective with the sixth grade student than with the fourth grade. She stated that "this process of thinking in dealing with numbers is more effective with children of greater maturity". This review of research yielded no study of adult estimation skills.

Related Variables

Results from studies relating intelligence and certain abilities with estimation or mental arithmetic indicate that students with higher intelligence tend to score higher on tests of estimation (Nelson, 1966) and use a greater variety of strategies than do students with lower intelligence (Brown, 1957). These results were further confirmed (Lawson, 1977) in a study of seventh graders. More specifically it was reported that students of better computational ability tend to be better estimators. Olander and Brown (1959) reported that performance in mental arithmetic was more dependent on general arithmetic ability than upon intelligence. Brown noted that high achievers in mental arithmetic use methods not usually employed by low achievers and that older pupils tend to use a greater variety of methods. A finding reported by

Olander and Brown (1959) was that boys performed better than girls on their mental arithmetic tests. Of the top 5 percent on scores of mental arithmetic across seven grade levels, 61 percent were boys and 39 percent girls.

Other abilities closely related to estimation and mental arithmetic have been found. For example, Paull (1971) observed that estimation of numerical computation was significantly correlated with problem solving, mathematical ability and verbal ability, and that the ability to compute rapidly was related to the ability to estimate numerical computation.

Guidelines Gathered From Research Review

This examination and synthesis of research provided the basis for several important decisions which guided the construction and use of the screening test used in this project:

1. An operational definition of computational estimation was necessary. Furthermore, assessment procedures should be commensurate with this definition.
2. Performance on computational estimation should be assessed through a visual mode that presented each item individually.
3. The amount of time for each item should be controlled to maintain assessment validity.
4. Open-ended items should be used with acceptable scoring intervals predetermined.

5. A comprehensive set of items should include a balance of operations, formats, contexts, and numbers.
6. Recovery items should be included to maintain student attention to the task.
7. Accelerated and high ability classes should be chosen to participate.

PROCEDURES

Introduction

This research was designed to identify and describe computational estimation processes used by good estimators and then to characterize these thinking strategies and estimation techniques. In order to accomplish this goal, several distinct tasks were done:

1. An estimation test to identify good estimators was developed.
2. Specific procedures for administering and scoring the estimation test were clearly identified.
3. An interview format, including appropriate problems and probes, was constructed.
4. A sample of subjects was selected.

The procedures used to complete each of these tasks will now be described.

Test Development

The first task involved the development of an estimation test. Such a test, called Assessing Compu-tional Estimation (ACE) was therefore created. The ACE served as the principal means of identifying good estimators and was developed from an earlier test that had been used to assess computational estimation skills (Bestgen et al., 1980). Pilot tests of the original version of the ACE that used different formats and times were administered to several levels of students as well as to experienced teachers of mathematics. Suggestions for improvement of the test were also solicited from both students and teachers. Pilot testing produced data that were used to guide development of a preliminary version of the ACE Test, which was subsequently scrutinized by three project consultants. Their suggestions and recommendations resulted in further refinement to produce the final version of the ACE Test (see pages 56 to 166).

Each of the 55 items on the ACE Test was produced on a 35mm slide with items shown sequentially using a carousel slide projector. This organization allowed for group administration and controlled the pace by allowing only a fixed amount of response time for each item. Care was taken to create straight computation items (those containing only numerical data) and applications items (those containing numerical data embedded in a real world context). These items provided a reasonable balance of whole numbers and decimals with only a few items involving fractions. This was done to avoid placing too much emphasis on fractions, since some research suggests that poor performance on estimation with fractions may be the result of a lack of understanding of fractions (Carpenter et al., 1980).

All of the 55 items (28 straight computation and 27 application) on the ACE Test were open-ended with answers written on a specially prepared answer sheet (see Appendix 1). The 6cm x 35cm answer sheet provided adequate space for the open-ended answers for the straight computation items on the front and application items on the back. It was purposely designed to be very compact to avoid any open space for students to either record the problem or do paper/pencil computation.

The open-ended format necessitated the construction of acceptable response intervals by the researchers. These intervals often reflected the results from the preliminary individual interviews and were designed to include answers obtained from various estimation strategies.

Timing for each item was determined by interviewing individual students and observing the time each required to respond. Using this information, time allotments for each item were identified. Since the ACE Test was developed to identify good estimators, it was decided that it would be better to allow too little rather than too much time. This rationale guided the final determination of the number of seconds allowed for each item. The response times of the items varied according to both the item and the grade levels. In administering the ACE Test, when the appropriated time for an item had lapsed, the slide with the estimation item was removed from view and replaced by a slide showing the number of the next item. This was displayed for 3 seconds, during which students recorded their answers to the previous problem. Thus, the time on each problem was carefully controlled.

The pilot testing of early versions of the ACE Test involved approximately 200 students per grade level in grades 6-12. The goal was to produce a valid test with good discrimination. In an item analysis of pilot results, items with a discrimination index of less than .30 were revised or discarded. Nearly all of the items used on the ACE Test showed a difficulty level between .30 and .60 during the field tests.

The pilot testing showed that due to the fast pacing and difficulty of the items some students lost interest or became frustrated during the test and did not perform up to their potential. Thus it was decided to insert some easier items periodically to renew student efforts. These "recovery items" (Computation problems 6, 12, and 18) were much easier than other exercises on the ACE Test and generally relied on mental computation.

Test-retest reliability for the final form of the

ACE Test used in this project ranged from .74 to .86 in grades 7 through 12.

Since testing was done at four different sites, it was important that directions and procedures be as uniform as possible across sites. All tests were administered by site directors, and in every case a uniform set of testing procedures was followed (see Appendix 2). No unusual administration problems were reported.

Selection of Sample

Results from pilot testing collaborated findings from other research in suggesting that the most likely candidates for good estimators would also be high achievers in mathematics. In an effort to characterize good estimators, observational data from the pilot testing and preliminary interviews were used to formulate some conjectures about good estimators. These hypothesized characteristics are shown in Table 1. An earlier version of a list of hypothesized characteristics was prepared during a meeting of consultants (see Appendix 3). This checklist was accompanied by discussion which led to the development of the hypothesized characteristics in Table 1. This research effort did not provide a systematic validation of these characteristics. However,

Table 1: Hypothesized Characteristics of Good Estimators

Quick with paper and pencil computations.

> Among the first to respond to oral questions and/or hand in their test papers.

Accurate with arithmetic computations.

> Check computations and strive for a high degree of accuracy.

Unafraid to be wrong.

> Risk contributing probable solutions to problems, easily cope with being wrong, and continue to probe for the solution.

Mathematical confidence.

> Possess good computational skills and realize potential to compute.

Demonstrated performance.

> Demonstrate adequate estimation skills and use them regularly.

Mathematical judgement.

> Judge a problem situation and determine when an estimate is appropriate and when an exact solution is needed.

Reasonableness of answers.

> Sense when an answer is not in the ballpark. Able to reject far out answers and seek more reasonable results.

Divergent thinking strategies.

> Have a knowledge of a variety of strategies and a tendency to search for alternate routes to a solution for a given problem.

this research did validate some of these constructs and additional discussion of them appears later within the summary of good estimators.

An examination of Table 1 suggests that if these characteristics are indeed valid, good estimators are likely to be high achievers. Therefore, when different ability classes were available, the upper level track classes were chosen to participate in the initial screening. Thus, seventh and eighth grade classes were typically accelerated classes, ninth and tenth grade classes were in algebra or geometry, and eleventh and twelfth graders in a fourth or fifth year of mathematics.

The adult sample was also select in that it was composed of members of a community service organization, some elementary and secondary mathematics teachers, and selected people from different professions including physicians, engineers, and business people. These adults were successful in their chosen occupation and nearly all of them were college graduates. Therefore, caution should be exercised in generalizing any of these findings to the general population.

Table 2 reports the number of subjects that participated in this research project.

Table 2. Frequency of Sample Subjects By Sex and Group

	Grade 7-8	Grade 9-10	Grade 11-12	Adult
Male	222	154	165	57
Female	209	205	126	49

Once classes were selected to participate, each teacher was sent a letter describing the purpose of the project and the extent of his involvement (see Appendix 6). One task for each teacher was to predict which of the teacher's students had good computational estimation skills. The list shown in Table 1 was given to the teacher to help in the selection process. Each teacher was asked to incorporate these criteria with his own knowledge of his students and identify those he predicted would do best on the ACE Test (see Appendix 7 for teacher recommendation form).

Test Scoring and Selection of Interview Sample

Upon completion of the test, each paper was scored using the acceptable ranges that had been established. The results were coded to expedite the analysis. In addition to background information, individual responses were coded into one of the following categories:

Within interval - any response within the predetermined acceptable interval that was not an exact answer. This was considered a correct response.

Exact - this was also treated as a correct response. It was reported separately only to provide additional insight into interpreting the data. In all cases the exact answer was within the predetermined acceptable interval.

Outside interval - any numerical value that was not within the predetermined acceptable interval. It was treated as an incorrect response.

No response - the answer was omitted.

This coding scheme provided not only a summary of all acceptable responses, but made it possible to identify the number of exact answers. This information was available as good estimators were identified and characterized. After all tests had been administered and scored at each site, the coded data were returned to the University of Missouri for complete analysis.

A listing of all scores obtained in each group was made and the cut-off level for the top 10 percent of the scores was established from these percentile ranks (see Table 9, page 50). The top 10 percent of each group then became candidates for follow-up interviews. When two or more people had the same total score, an examination of the number of exact and number of acceptable answers was made, and the subject with the least number of exact answers was the preferred interview candidate. Consultation with each teacher about his students' willingness to cooperate and the students' availability further influenced the final selection of the interviewees.

Interview Development

The interview provided the means of learning what strategies and processes the students used in solving different estimation problems. Since students were available for a limited time, usually one class period, only a few estimation problems could be posed. The following describes briefly the interview problems; but the entire interview packet may be found in Appendix 4.

SEGMENT I: Straight Computation. A cluster of straight computation estimation problems that appeared in the ACE Test was included in the interview. It was anticipated that insight could be gained by posing these same problems in the interview to identify specific strategies used to solve them on the ACE Test. Furthermore, using a subset of the screening test provides information related to the consistency of response, although the results may be confounded by the time factor. Very strict time controls were in effect during the screening test while no time limit was imposed during the interview.

STRAIGHT COMPUTATION INTERVIEW PROBLEMS

1. $$\begin{array}{r} 87\,419 \\ 92\,765 \\ 90\,045 \\ 81\,974 \\ +\ 98\,102 \\ \hline \end{array}$$

2. $31 \times 68 \times 296$

3. $8\,127\overline{)474\,257}$

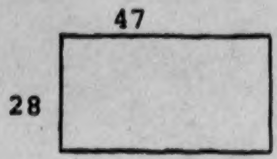
4. $\frac{347 \times 6}{43}$

5. $1\frac{7}{8} \times 1.19 \times 4$

SEGMENT II: Application. Estimation is an important skill because of its utility and practicality in everyday situations. Techniques used when estimating as well as the degree of accuracy often depend upon the situation encountered. In order to document differences in strategies used when the context differs, applied estimation exercises were given in the interview. Although an estimation exercise within an applied context requires some problem solving, the exercises were designed to minimize the need for complex problem-solving skills. Problems were designed which lent themselves to several different strategies so that insight into the most popular strategies used could be obtained. These applied problems are presented here.

APPLICATION INTERVIEW PROBLEMS

6. About how much area does this rectangle have?



7. If 30% of the fans at the 1979 Superbowl bought one soda, about how many sodas were sold at that game?

1979 attendance:
106,409

8. At the 1979 Superbowl 8,483 hotdogs were sold for $.60 each. About how much resulted from selling the hotdogs?

9. Here are 3 estimates for the total attendance for the past 6 Superbowl games:

1 000 000
600 000
550 000

Which is the best estimate?

Year	Attendance
1974	73 655
1975	86 421
1976	91 943
1977	96 509
1978	93 421
1979	106 409

10. The 1979 Superbowl netted $21 319 908 to be equally divided among the 26 NFL teams. About how much does each team receive?

11. (NOTE: The student was instructed to respond to this item using an actual menu. See Appendix 3).

Three people have dinner. They order:

Bacon n Cheese Steakburger Platter	$2.89
Super Steakburger Platter	$2.64
Chili-Mac	$1.47
2 small coca-cola's	45¢ each
1 hot chocolate	35¢
1 hot pie	76¢

About how much money will be needed to pay the bill?

12. The Thompson's dinner bill totaled $28.75. Mr. Thompson wants to leave a tip of about 15%. About how much should he leave for the tip?

13. Which carton has more soda?

COKE	PEPSI
6 bottles	8 bottles
32 oz. each	16 oz. each
$1.79	$1.29

14. Which soda is the cheapest?

15. This is a grocery store ticket which has not yet been totaled. Estimate the total.

0.79	AGr
0.79	AGr
0.44	AGr
1.30	APr
0.34	APr
1.05	AGr
0.57	AMt
0.29	AGr
3.65	AGr
0.30	AGr
0.31	AGr
2.29	AGr
0.11	APr
0.34	APr
0.08	AGr

SEGMENT III: Calculator Segment. Estimation skills are essential when using calculators to prevent acceptance and use of unreasonable answers. The final portion of the interview asked sutdents to make estimates and then to compare the accuracy of their estimates with results obtained from using a calculator. The calculator had been previously programmed to make systematic errors in computing. Students were observed as they used this calculator to determine how sensitive they were to the systematic errors. This segment of the interview provided an unobtrusive measure of confidence in their estimate and/or their willingness to challenge the calculator output. A full description of this segment is presented here.

CALCULATOR SEGMENT

Directions to Interviewer: Program the calculator being used to make a consistent error in computing each answer - first stage: 10% error (above actual answer) - second stage: 25% error (above actual answer) - third stage: 50% error (above actual answer).

Directions to Student: "You seem to have developed some very fine estimation skills. You've done an excellent job both on the test I gave you in class and in the questions I've asked you individually. Up to this point I have asked you to estimate in a variety of situations but haven't told you how accurate your estimates have been. In this last task, I'd like for you to estimate the answer to a few computation problems, then compare your estimate with the calculator result. Let's see how accurate your estimates are."

Directions to Interviewer: Present each exercise individually, answer questions the student asks but do not start probing until stage 2 unless the error is noticed. At stage 2, use only the probes suggested. Probes for stage 3 are outlined but you may vary from these as seems appropriate.

	Exercise	Probes
Stage 1:	436 + 972 + 79	
	42 963 ÷ 73	
	896 x 19	
Stage 2:	28 x 47	"Tell me how you got your estimate."
	896 + 501 + 789	Ask student before calculator is used: "Do you think the actual answer will be above or below your estimate?"
Stage 3:	22 x 39	Ask student before calculator is used: "Can you give me a better estimate?"
	252 x 1.2	"How sure are you of your estimate?" "Could you give me an upper bound for an estimate for this problem?"
When student notices error, ask:		"Why do you think that? Perhaps you made an error in keystroking." (Let them verbalize error.) "When did you notice that the calculator was making an error? - Why didn't you tell me about it at that point?"

SEGMENT IV: Attitude/Concept Questions. Some questions were also designed to learn about the students' concept of estimation. These questions included:

* Do you remember the estimation test you took in class? Do you ever do this kind of thing in or out of school? In mathematics class? (At work? As a consumer?)

* Do you think estimation is part of mathematics?

* Do you think estimation is an important skill?

* What is estimation?

* Where did you learn to estimate?

* Do you practice estimating?

* If you were going to give someone hints on good ways to estimate or good strategies to use when estimating, what would you tell them?

To supplement the interview problems, specific probes were developed to provide consistency among interviewers as well as to more carefully focus on specific characteristics hypothesized to be common among good estimators. For example, on several problems, students were asked to describe their degree of confidence in their estimates (using a five-point semantic differential scale). A list of standard probes used in all interviews is described in Appendix 4.

Approximately 35-50 minutes were required for each interview. Identical notebooks contained the complete set of problems and specific probes to be used. These specific probes were considered to provide the common data base across all sites. However, each interviewer was free to ask additional questions and initiate probes as seemed appropriate.

All interviews were tape-recorded and relevant data transferred to the Interview Summary Sheet (Appendix 5). Selected portions of interviews were transcribed to aid in describing specific estimation strategies. This provided the data base for all the interviews. A 25-minute video tape of portions of one adult interview was made. Anyone interested in viewing this 3/4 inch cassette tape should contact the principal investigator.

ACE RESULTS

By Individual Problems

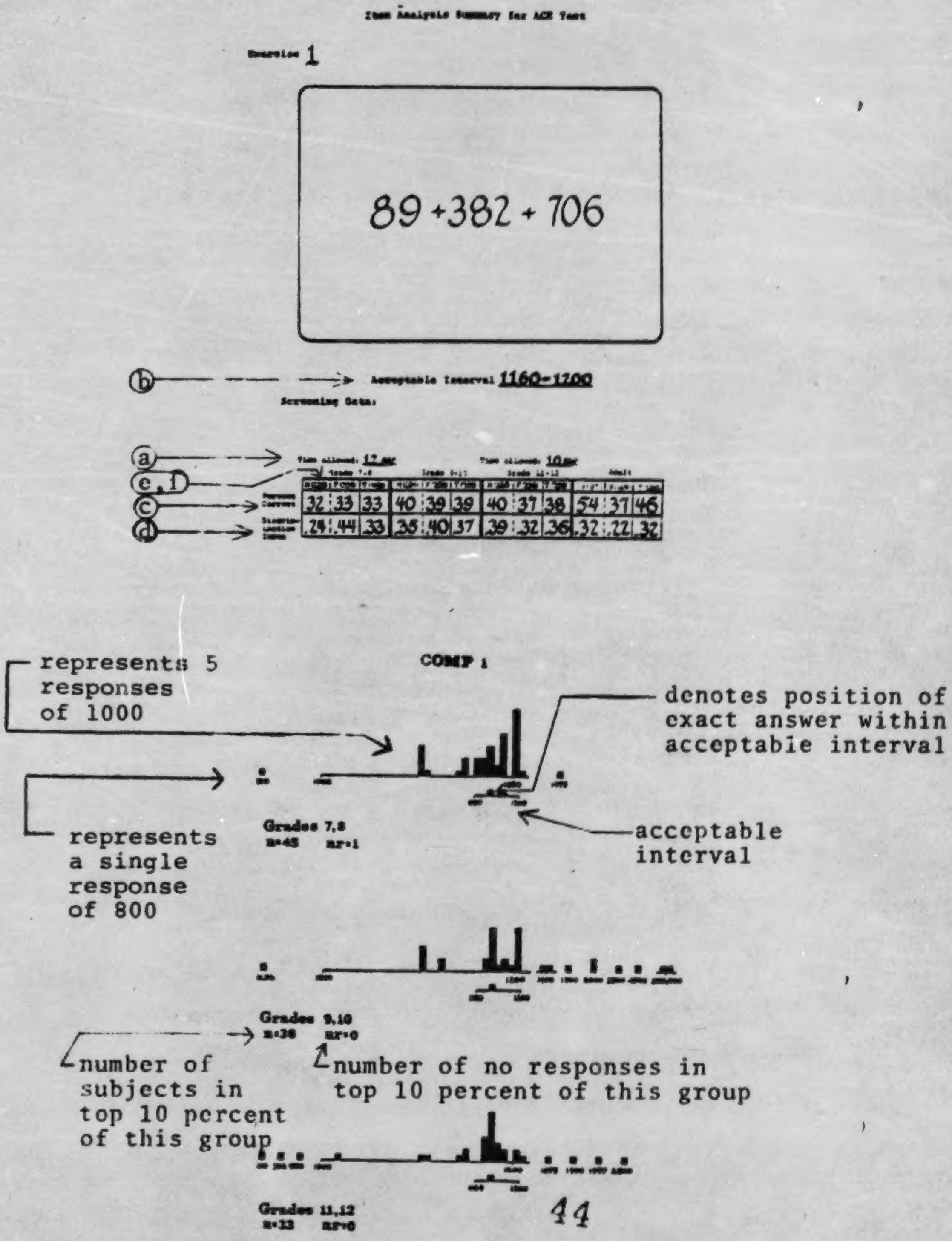
The primary data source of this research project is the interview data obtained from good estimators. However, the procedure used in selecting these good estimators - the use of the ACE Test as a screening device - involved over 1100 people and resulted in some informative data. Due to the nature and scope of this project, a unique data base was accumulated and will be discussed briefly.

The compiled data for each item on the ACE Test are reported at the conclusion of this section on the buff colored pages 56 to 166. Included with each item are the following statistics:

a) amount of response time provided each group,
b) acceptable scoring interval,
c) difficulty for each group,
d) discrimination index within each group,
e) difficulty by sex for each group, and
f) discrimination index by sex for each group.

All this information for each item appears on one page. On the facing page is a graphical representation of the estimates given by those scoring in the top ten percent in the three student groups. Table 3 illustrates these facing pages and includes explanatory notes for the various results reported. The researchers had some trepidation

Table 3. Explanatory Notes of Summary Data Provided for ACE Test



Item Analysis Summary for ACE Test

Exercise 1

89 + 382 + 706

ⓑ → Acceptable Interval 1160-1200

Screening Data:

ⓐ → Time allowed: 12 sec   Time allowed: 10 sec

(e, f) ↗

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
ⓒ → Percent Correct	32	33	33	40	39	39	40	37	38	54	37	46
ⓓ → Discrimination Index	.24	.44	.33	.35	.40	.37	.39	.32	.36	.32	.22	.32

COMP 1

represents 5 responses of 1000

denotes position of exact answer within acceptable interval

represents a single response of 800

acceptable interval

Grades 7,8
n=45   nr=1

Grades 9,10
n=38   nr=0

number of subjects in top 10 percent of this group

number of no responses in top 10 percent of this group

Grades 11,12
n=33   nr=0

in establishing the acceptable intervals and were concerned that "close answers" not in the acceptable interval might occur. The graphs indicate that the intervals included most of the reported estimates and that none of the intervals were too restricted to exclude many close estimates. This graph also provides additional insight into the range of responses that were made by the good estimators.

No data from adults are included in the graphs because of the small number of adults included in the screening. Nevertheless, the responses on a sample of problems from the top ten percent of the adult group are consistent with the results shown graphically for the student groups.

Considerable space is devoted to presenting this summary of individual problems and a thorough examination of it is time consuming. Nevertheless, these data provide the basis for much interesting exploration that transcends the purpose of this project. In addition to the sex and age differences reported, comparisons between types of numbers (whole number versus decimals), comparisons among operations (addition, subtraction, multiplication and division), and comparisons between formats (straight computation versus applied problems) could all be examined. Although readers are encouraged to make these and other comparisons, only one comparison, that which involves different formats, will be discussed.

Will performance be different if the same numerical information is presented in a straight computation or in an applied context? It was hypothesized that placing the numbers in an applied context would improve performance even though it required the subjects to process the information given, choose the appropriate operations and then formulate the estimate. All of this was to be done within the same time constraints that were used for the straight computation items. It was felt that a context provides a basis for placing reasonable bounds on the degree of exactness required for a given item, which would in turn aid the estimation process. It was also felt that such items would be more appealing than those that required response to abstract numerical data and would improve attention, concentration and dedication to the task. Observations during the administration of the ACE Test supported this conjecture. For example, interest and attention increased when the application items were shown. Furthermore upon completion of the test, many subjects claimed to have liked the application portion better and felt they had done better on it. An examination of the results of the ACE Test showed many fewer "no responses" on the application than on the straight computation portion.

The results from three pairs of parallel items in the ACE Test are summarized in Table 4. Perhaps the most noteworthy observation is the lack of a consistent pattern of performance across these three parallel items. For example, in items C23 and A20 there is a marked difference in performance, with higher performance in the application context for all levels and both sexes. In items C8 and A24, no consistent differences in performance were observed, but in C24 and A23 there were slight differences in favor of the straight computation for all levels and both sexes. This result was surprising, but an examination of C24 and A23 shows the latter item presented several pieces of information for the subject to process before making an estimate. Thus a plausible explanation for these differences is that the application item required much heavier problem-solving demands. Therefore, the value of the applied context in making estimates seems to rest heavily on the amount of data presented as well as the level of problem solving required.

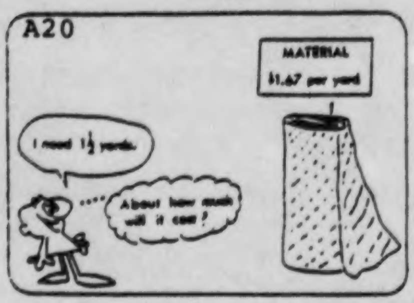
The final item described here deals with the subjects' self evaluation of their ability to estimate. The question

Table 4. Results Reported in Percents on Parallel Items Included on the Computation and Application Portions of the ACE Test

Problem Computation Application

C23 1⅓ x 1.67

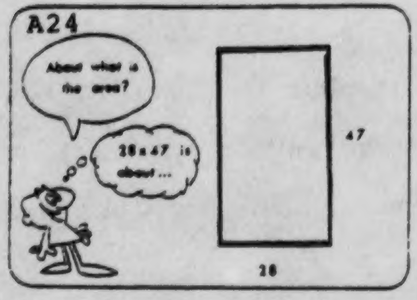
vs.


A20
MATERIAL
$1.67 per yard
I need 1⅓ yards.
About how much will it cost?

	Computation				Application			
	Grade				Grade			
	7,8	9,10	11,12	Adult	7,8	9,10	11,12	Adult
M	28	54	44	67	57	75	71	84
F	19	39	33	53	50	75	71	78

C8 28 x 47

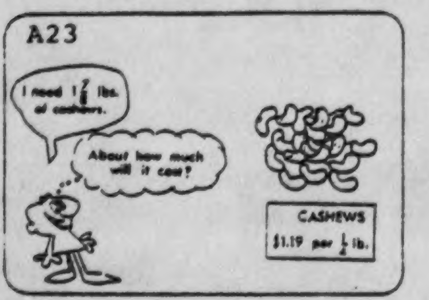
vs.


A24
About what is the area?
28 x 47 is about...
47
28

	Computation				Application			
	Grade				Grade			
	7,8	9,10	11,12	Adult	7,8	9,10	11,12	Adult
M	28	33	36	68	21	40	36	61
F	24	36	33	49	16	28	32	51

C24 $1\frac{7}{8}$ x 1.19 x 4

vs.


A23
I need 1 7/8 lbs. of cashews.
About how much will it cost?
CASHEWS
$1.19 per ½ lb.

	Computation				Application			
	Grade				Grade			
	7,8	9,10	11,12	Adult	7,8	9,10	11,12	Adult
M	18	37	38	58	18	29	30	35
F	17	25	22	35	14	15	21	37

"Are you a good estimator?" was the final item on the ACE Test. At the time this question was asked, the subjects had just completed estimating answers to 55 problems under tight time restrictions. This information may add insight to the results included in Table 5.

It is interesting that even though these groups were above-average achievers, in no group did a majority answer "yes" to the question "Are you a good estimator?" In fact, the affirmative responses for all inschool groups were consistently low, ranging from 10 percent for the seventh and eighth graders to 14 percent for eleventh and twelfth graders, with nearly one-third of the adults answering "yes". Even more striking is the consistently higher ratio of "yes" responses of males to females. This ranged from about 3:1 for seventh-eighth graders to 5:1 for ninth-tenth graders. For example, 56 percent of the seventh-eighth grade females reported "not sure" to the question "Are you a good estimator?" and in all other groups the majority of the females said "no".

It was conjectured that subjects rating themselves as good estimators would do better on the ACE Test than subjects who said "no" or "not sure". The results reported in Table 6 seem to support this conjecture. The means of

Table 5. Summary of Responses* to Question: "Are you a good estimator?"

		GRADE 7 - 8			GRADE 9 - 10			GRADE 11 - 12			ADULT		
		YES	NO	NOT SURE	YES	NO	NOT SURE	YES	NO	NOT SURE	YES	NO	NOT SURE
TOTAL GROUP	TOTAL	10	34	56	11	45	43	14	38	47	32	39	26
	MALE	14	29	57	20	23	55	19	25	53	49	21	24
	FEMALE	5	39	56	4	62	34	6	56	37	12	59	29
TOP 10 PERCENT	TOTAL	23	12	65	27	19	54	31	11	57	73	7	20
	MALE	30	7	63	36	4	60	36	5	59	73	9	18
	FEMALE	8	23	69	8	50	42	23	23	54	75	0	25

*Reported in percents

subjects rating themselves as good estimators were typically 2 to 4 points higher than the other two groups of subjects. Consistent results were found across all four groups. Also, in every case males were not only more likely to claim to be good estimators than females, but males who said they were good estimators scored higher on the ACE Test than did their female counterparts. This observation was confirmed in Table 6 and further supported by Table 10 which reports a disproportionate percent of males to females in the top 10 percent of the ACE Test for all four groups. These results held for both the computation and application portions of the ACE Test with the single exception of the adults. This exception is probably best explained by the fact that only five female adults answered "yes" to the question "Are you a good estimator?" and this group was compared to 26 males who also said "yes" to the question.

These results provide another perspective of subject self-confidence and confirm that subjects who thought they were good estimators were indeed much better than other subjects. These results provide strong evidence that even though these were high achieving students in mathematics, they did not perceive themselves as good estimators. The self assessment of adults was more posi-

Table 6. Summary of Mean Scores on ACE Test and Subject Self Appraisal of Estimation Skills

		GRADE 7-8			GRADE 9-10			GRADE 11-12			ADULT		
		YES	NO	NOT SURE	YES	NO	NOT SURE	YES	NO	NOT SURE	YES	NO	NOT SURE
TOTAL	C	12.5	8.1	10.7	15.8	12.5	14.7	16.1	12.3	14.9	18.6	13.3	16.4
	A	15.0	9.2	11.9	16.9	13.7	16.3	18.4	12.9	16.8	21.8	16.1	19.8
MALE	C	13.3	8.4	11.4	16.6	12.7	15.6	16.7	11.5	15.2	18.4	16.8	17.3
	A	16.3	9.4	13.0	17.2	13.6	17.3	18.7	12.0	17.0	21.6	17.4	20.2
FEMALE	C	10.3	7.8	9.9	13.1	12.4	13.6	13.8	12.7	14.3	19.8	11.7	15.6
	A	10.8	9.0	10.7	15.9	13.7	15.1	17.1	13.5	16.2	23.2	15.5	19.4

C denotes computation score

A denotes application score

tive, but even the, only one-third of the adults claimed to be good estimators. Furthermore, across all groups there was a marked sex difference in subjects' perception of themselves as good estimators.

For All Subjects

The selection procedure resulted in a data base on the ACE Test that is higher than would normally be expected within each of the four groups. For example, a subsequent comparison of the ninth grade classes participating in this project with all ninth graders in the Hazelwood School District (an upper middle class suburb of St. Louis) confirm this this suspicion for the inschool groups (see Table 7). Therefore, all

Table 7. Comparison of Results on ACE Test Between Students Screened in this Research Project and a Cross Section of Ninth Graders in the Hazelwood School District.

	Hazelwood N=360	Research N=359
Computation Portion	m = 8.8 S.D.= 4.0	m = 13.8 S.D.= 4.4
Application Portion	m = 11.3 S.D.= 5.9	m = 15.1 S.D.= 5.2

phases of interpretation should keep in mind that the performance levels reported in this research reflect a sample that would score above general population levels.

A summary of some results on the two parts of the ACE Test is presented in Table 8. These results highlight performance in several different ways and reveal some interesting contrasts.

Table 8. Mean Scores on ACE Sub-Tests

Computation

	Grade 7-8	Grade 9-10	Grade 11-12	Adult
Female	9.0	12.9	13.4	13.6
Male	10.7	15.0	14.4	17.4
Total	9.9	13.8	14.0	15.6

Application

	Grade 7-8	Grade 9-10	Grade 11-12	Adult
Female	9.9	14.3	14.8	17.4
Male	12.3	16.3	15.9	19.3
Total	11.2	15.1	15.4	18.4

In particular, an examination of the data in Table 8 suggests several trends:

1. Performance increases consistently across all

age groups. Since instruction designed to develop estimation skills is not typically found in the secondary school, it appears that additional number experiences and opportunities to estimate in real world situations improve performance in estimation. This seems to be a reasonable conjecture and is further supported by the improvement in adult scores over eleventh and twelfth grade students' scores, especially on the applied portion of the ACE Test.

2. There were marked sex differences among all four groups of subjects and across both types of estimation items. In every case males performed higher than females.
3. There was generally higher performance on estimation items in an applied context than on straight computational items. These differences were observed across sex and age levels.

These trends provide a basis for further exploration. In addition to these global observations, a careful examination of the statistics reported for individual items on pages 56 to 166 will provide additional insight and perhaps suggest other trends worthy of investigation.

The main purpose of the ACE Test was to identify good

Table 9. Percentile Ranks of Total Scores on ACE Test for Each Group.

NUMBER OF CORRECT RESPONSES	GRADE 7 - 8	GRADE 9 - 10	GRADE 11 - 12	ADULT
2	1			
3	2			
4	4			
5	6			
6	9	↑		
7	11			
8	15	1		1
9	18		1	↑
10	20	↓	2	2
11	23	2	↓	↓
12	27	3	3	3
13	30	4	↓	4
14	33	6	4	5
15	36	7	6	↑
16	38	9	9	
17	40	12	11	7
18	43	14	13	↓
19	46	16	15	8
20	48	19	17	9
21	50	21	20	↑
22	51	25	23	12
23	55	27	26	↓
24	61	30	30	16
25	64	35	33	20
26	67	39	37	↓
27	71	43	42	24
28	73	47	46	25
29	76	50	50	33
30	78	55	54	37
31	80	58	58	↓
32	84	64	63	39
33	86	69	66	42
34	88	73	69	51
35	90	76	72	54
36	92	79	75	57
37	94	82	80	61
38	95	85	81	65
39	97	87	84	68
40	97	89	88	73
41	98	91	92	75
42	↑	93	94	78
43	99	95	96	80
44	↓	96	97	83
45	100	98	98	86
46		↑	99	90
47			100	94
48		99		↑
49		↓		
50		100		98
51				↓
52				99
53				↓
54				100

estimators. The difficulty levels attained and strong positive discrimination indices account for the sizeable variance and spread of subject scores on the ACE Test. Table 9 summarizes a distribution of percentiles for various raw scores on the ACE Test for each of the four groups. The total range was from 0 to 54 and sizeable ranges are noted within each group. The broken lines denote the cut-off for the top ten percent of each group.

For Top 10 Percent of Subjects

Those who scored in the top ten percent in each group of the ACE Test formed the pool of subjects from which interviewees were later selected. Table 10 shows that many more males were included in the top ten percent

Table 10. Distribution by Number and Percent of Subjects Scoring in Top 10 Percent on the ACE Test

GROUP	MALE	FEMALE	TOTAL
Grade 7 - 8	30 70%	13 30%	43
Grade 9 - 10	25 68%	12 32%	37
Grade 11 - 12	22 63%	13 37%	35
Adult	11 73%	4 27%	15

than females in each of the four groups. Consequently the interviews included a disproportionate number of males and this fact should be kept in mind when interpreting these results.

It is not surprising that subjects who scored in the top ten percent on the ACE Test responded very differently to the question "Are you a good estimator?" than did the whole group screened, as an examination of Table 6 shows. In particular, Table 6 shows that about one-quarter of these inschool subjects and nearly three-fourths of the adults thought they were good estimators, levels which are much higher than similar responses for the total subjects in each of the groups. Table 6 also shows that sex difference responses to this question were marked among the top ten percent.

Five of the straight computation items from the ACE Test were also included in the interview battery. The intent was to obtain a measure of consistency of responses and to identify the strategies used to arrive at the estimate during the interview. Unfortunately, the consistency of performance on these problems was confounded by the testing conditions. In particular, the time constraints used with the ACE Test did not operate during the interview in which subjects had as much time as they wanted. Therefore, different strategies as well as additional checking and refinement may have been employed during the interview. In fact, later interview data show many fewer errors were made on these problems from the ACE Test when they appeared as interview questions.

Table 11 provides an overview of these comparisons for the inschool subjects. These data were obtained by comparing the subject's response from the ACE Test with the response on the same interview problem. Thus in problem C3, Table 11 shows that seven of the 16 ninth and tenth graders reported the same answer, whereas nine reported different answers, six of which had been incorrect on the ACE Test.

An analysis of Table 11 also confirms that the majority of students in each group obtained different estimates in the interview than on the ACE Test. This finding, together with the much higher percentage of correct responses during the interviews, makes it very risky to claim that the same strategies used during the interview were also applied during the ACE Test. Therefore, subsequent discussion of the interview data will be limited to responses and strategies collected through the interview.

Table 11. A Comparative Analysis of the Student Responses to Five Problems Which Appeared on Both the ACE Test and the Interview.

		Grade	Frequency of Responses		
			same*	diff/corr**	diff/incorr***
Problem C3	87 419 92 765 90 045 81 974 + 98 102	7 - 8	4	0	4
		9 - 10	7	3	6
		11 - 12	5	3	13
Problem C15	31 x 68 x 296	7 - 8	0	2	6
		9 - 10	3	1	12
		11 - 12	2	3	15
Problem C9	$8\,127\overline{)\,474\,257}$	7 - 8	1	2	5
		9 - 10	1	5	10
		11 - 12	3	7	11
Problem C16	$\frac{347 \times 6}{43}$	7 - 8	3	2	3
		9 - 10	1	5	10
		11 - 12	3	10	8
Problem C24	$1\frac{7}{8} \times 1.19 \times 4$	7 - 8	2	5	1
		9 - 10	6	6	4
		11 - 12	3	11	8

* Same response on parallel ACE item and Interview problem.

** Correct response to ACE items but different than response to parallel interview problem.

*** Incorrect response to ACE item and different than response to parallel interview problem.

Exercise 1

89 + 382 + 706

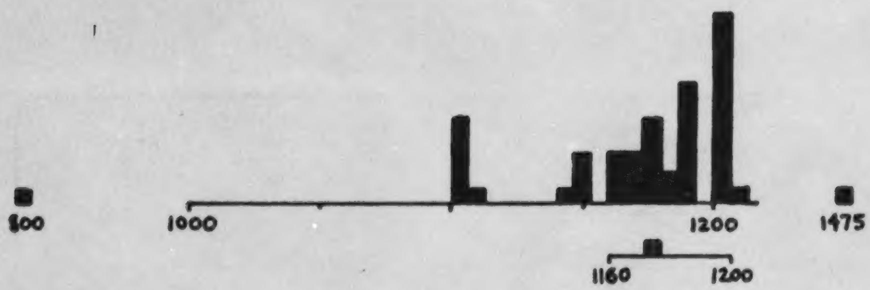
Acceptable Interval 1160-1200

Screening Data:

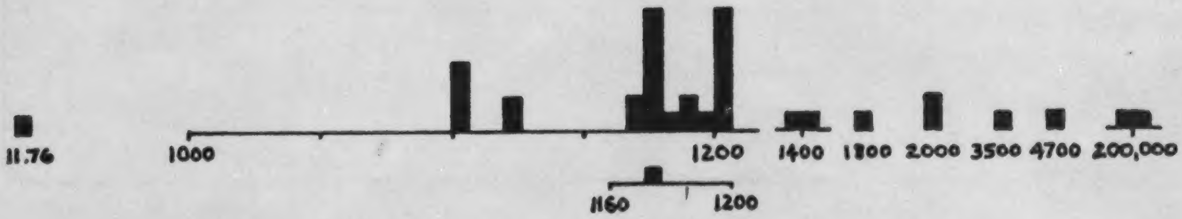
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	32	33	33	40	39	39	40	37	38	54	37	46
Discrimination Index	.24	.44	.33	.35	.40	.37	.39	.32	.36	.32	.22	.32

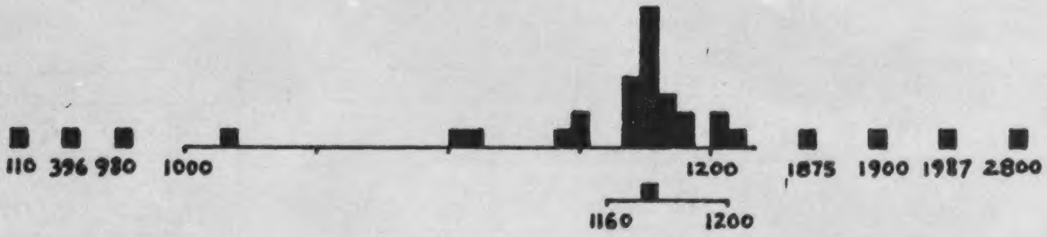
COMP 1



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 2

7465 - 572

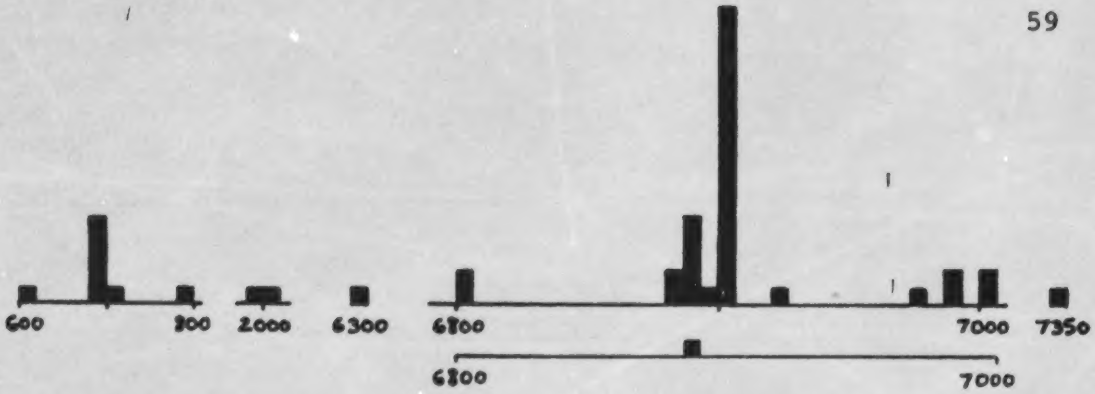
Acceptable Interval 6800-7000

Screening Data:

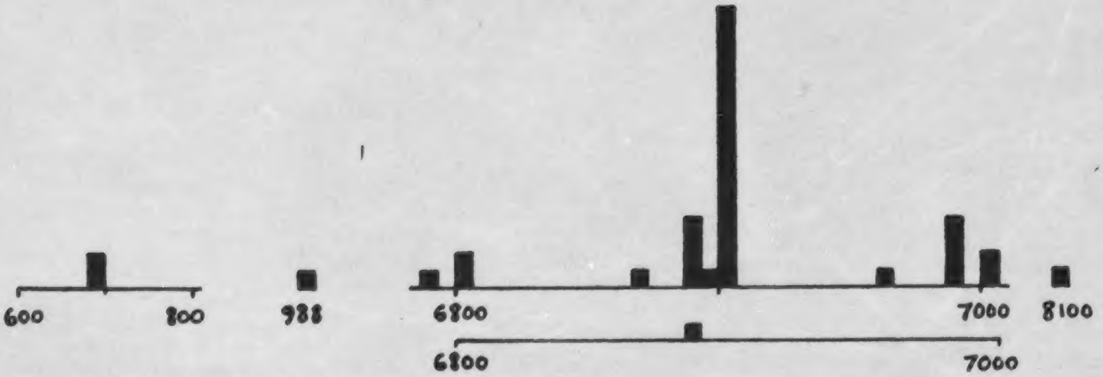
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7 -8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	45	35	40	63	51	56	57	56	57	61	45	54
Discrimination Index	.45	.51	.49	.38	.36	.38	.42	.34	.38	.20	.43	.34

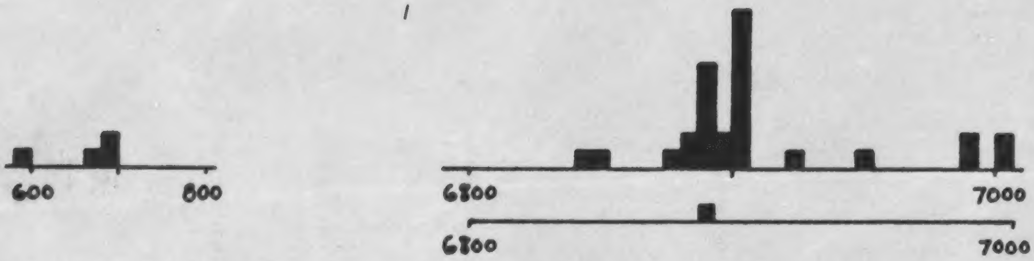
COMP 2



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise 3

$$\begin{array}{r} 87,419 \\ 92,765 \\ 90,045 \\ 81,974 \\ +\ \underline{98,102} \end{array}$$

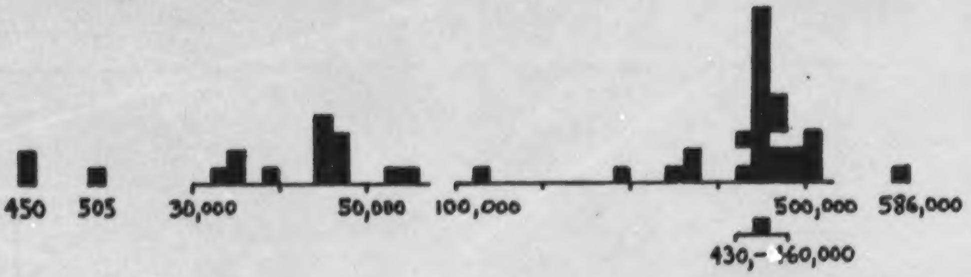
Acceptable Interval 430,000-460,000

Screening Data:

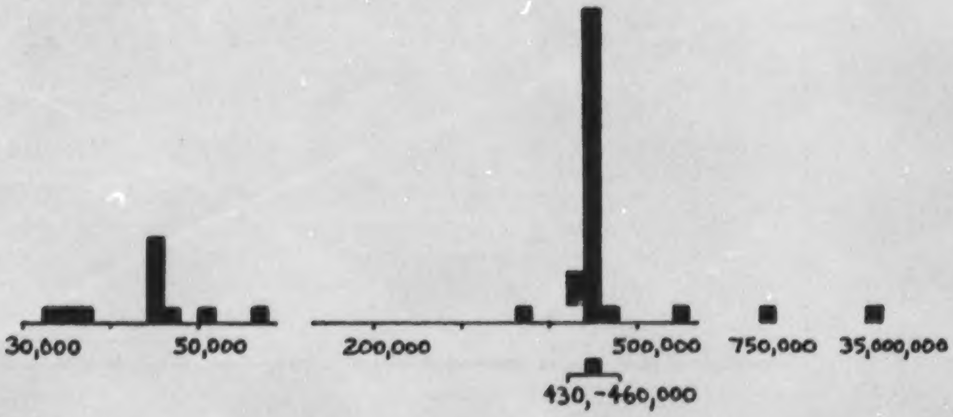
Time allowed: 17 sec. Time allowed: 15 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	10	2	6	21	10	15	12	10	11	40	22	32
Discrimination Index	.41	.09	.32	.47	.40	.45	.21	.34	.26	.44	.47	.48

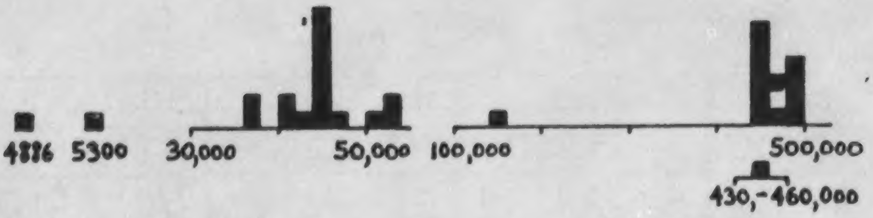
COMP 3



Grades 7,8
n=45 nr=2



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=2

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 4

37,689 - 18,812

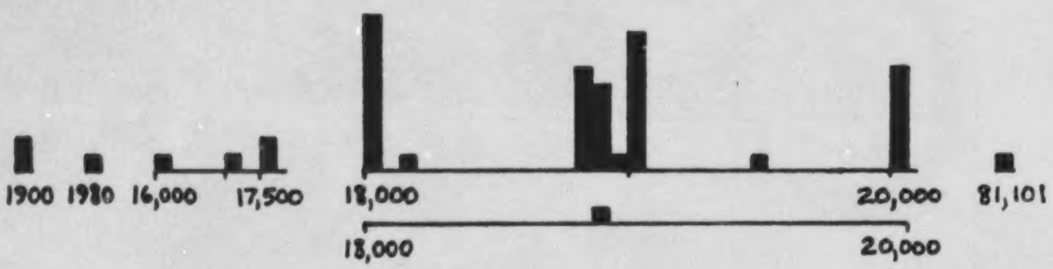
Acceptable Interval 18,000-20,000

Screening Data:

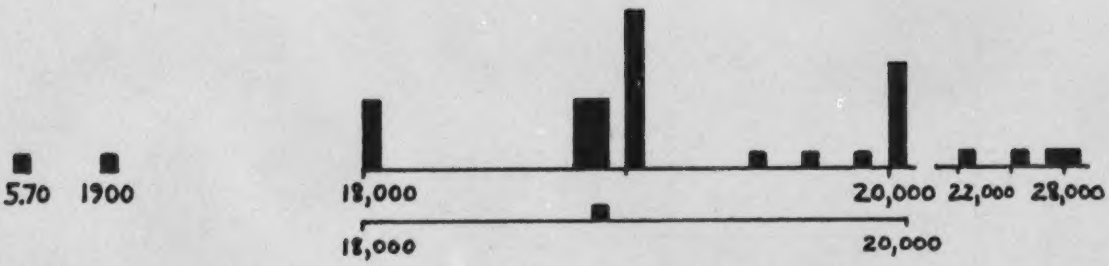
Time allowed: 12 sec. (Grades 7-8, 9-10) Time allowed: 10 sec. (Grade 11-12, Adult)

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209	T (431)	M (154)	F (205)	T (359)	M (165	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	36	25	30	68	44	54	56	52	54	61	63	62
Discrimination Index	.54	.54	.54	.52	.38	.48	.51	.42	.47	.44	.55	.45

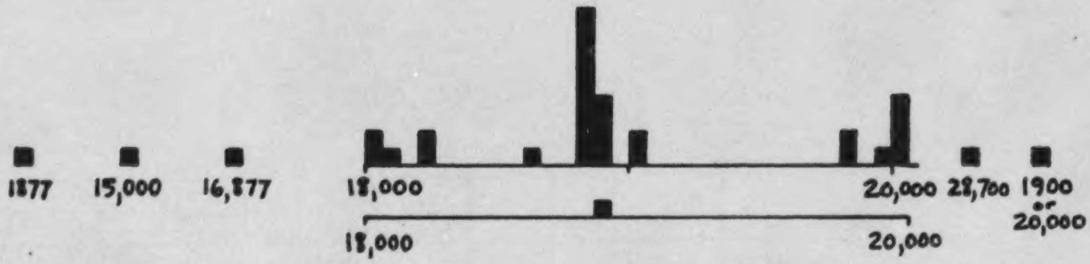
COMP 4



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 5

$$87 \times 62$$

Acceptable Interval 4800-6000

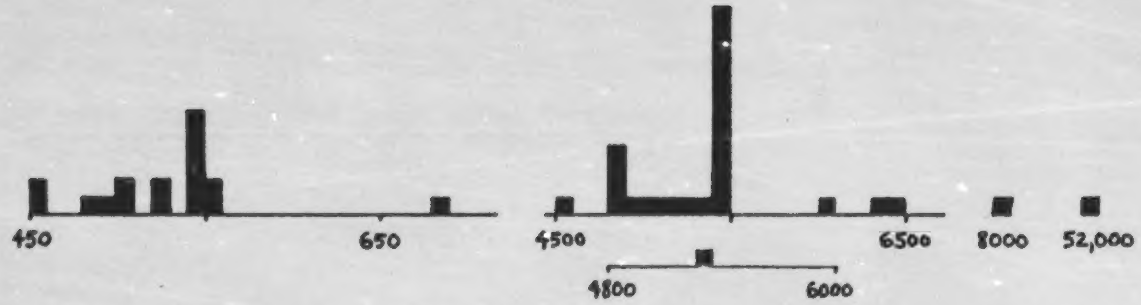
Screening Data:

Time allowed: 12 sec

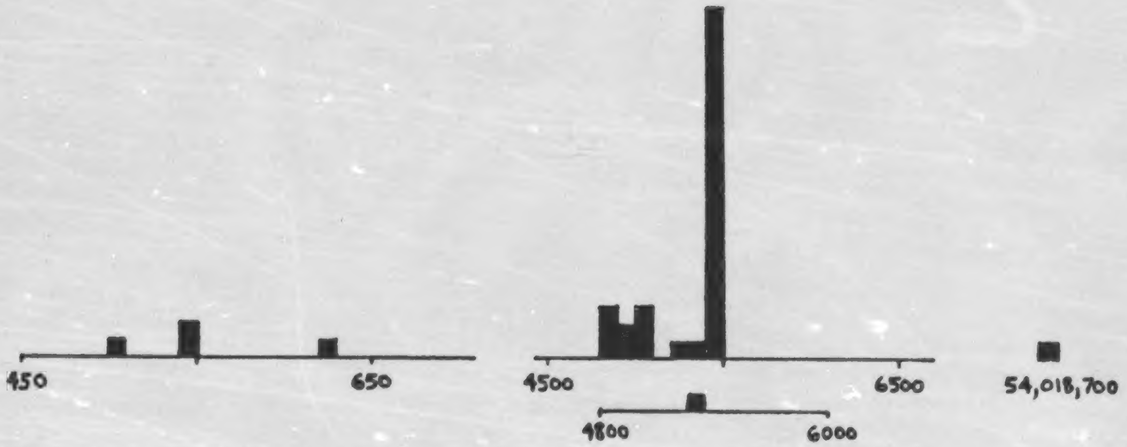
Time allowed: 10 sec

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	26	29	27	45	40	42	38	37	38	40	53	46
Discrimination Index	.33	.50	.40	.45	.45	.45	.39	.52	.44	.62	.42	.45

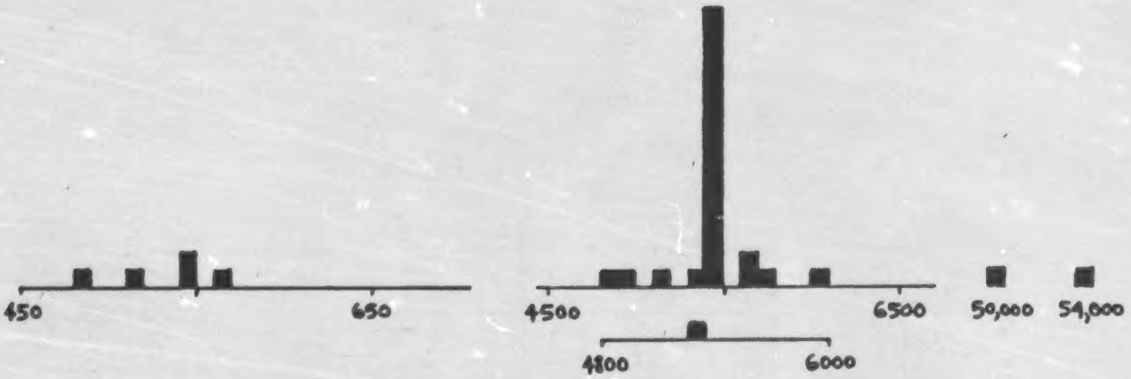
COMP 5



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=2

Note: ■ represents one response

Exercise 6

$$50 + 200 + 6$$

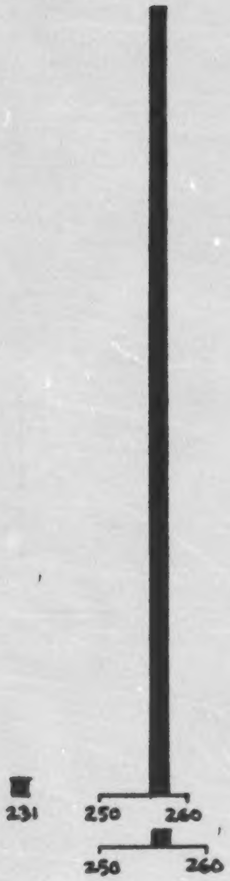
Acceptable Interval 250-260

Screening Data:

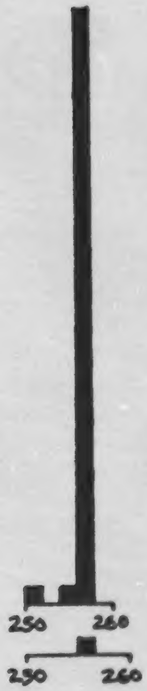
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	93	89	91	98	98	98	96	98	97	91	94	92
Discrimination Index	.30	.31	.31	.17	.18	.17	.18	.03	.11	.10	.28	.15

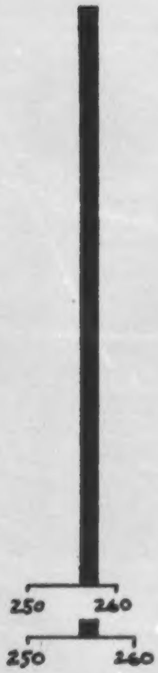
COMP 6



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 7

415 × 7

Acceptable Interval 2700-3000

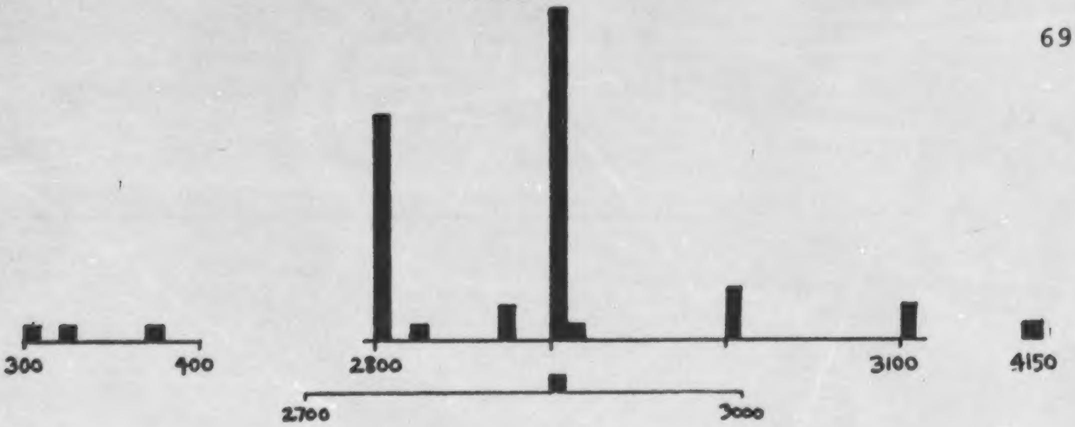
Screening Data:

Time allowed: 12 sec.

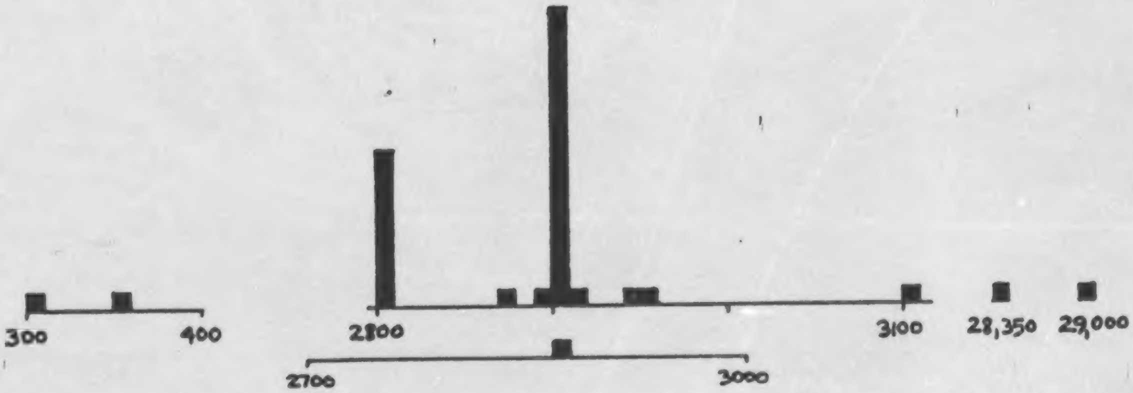
Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	65	72	68	73	77	75	68	75	71	75	67	72
Discrimination Index	.35	.35	.33	.23	.14	.16	.21	.21	.20	.11	.17	.16

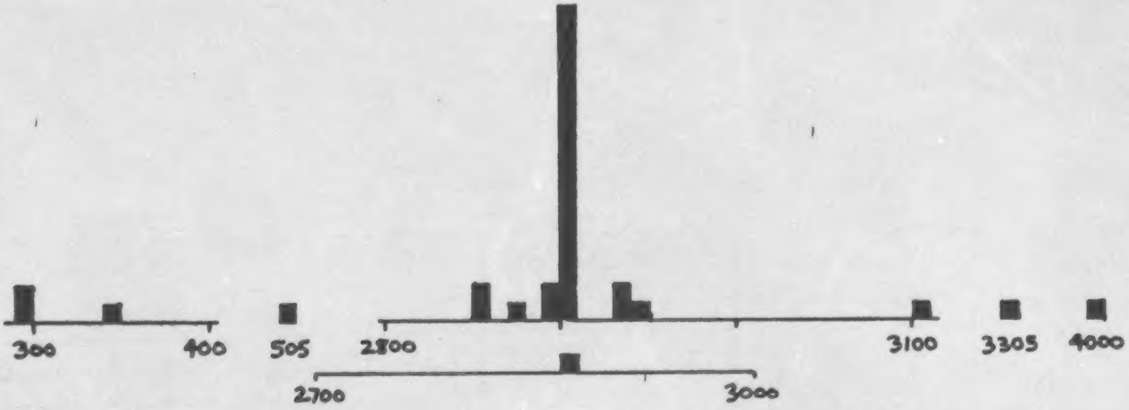
COMP 7



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 8

$$\begin{array}{r} 28 \\ \times\ 47 \\ \hline \end{array}$$

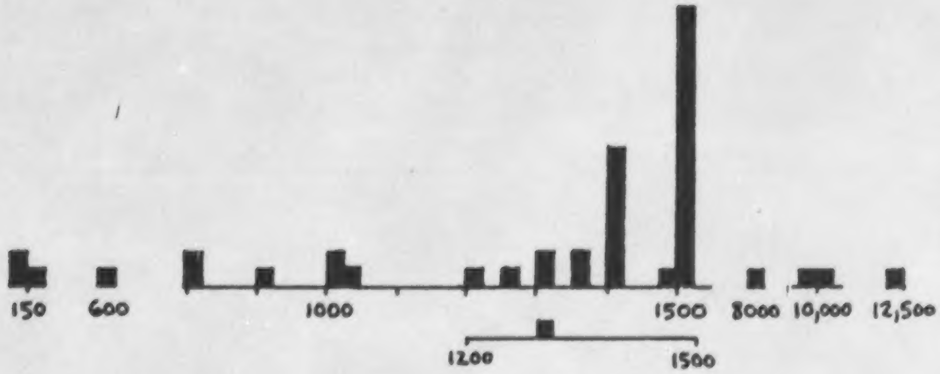
Acceptable Interval 1200~1500

Screening Data:

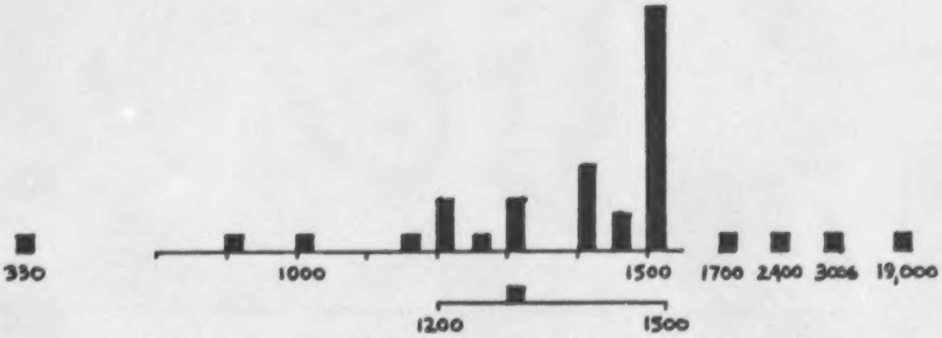
Time allowed: 12sec Time allowed: 10sec.

	Grade 7-8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	28	24	26	33	36	35	36	33	35	68	49	59
Discrimination Index	.50	.47	.49	.45	.44	.43	.39	.41	.40	.54	.36	.49

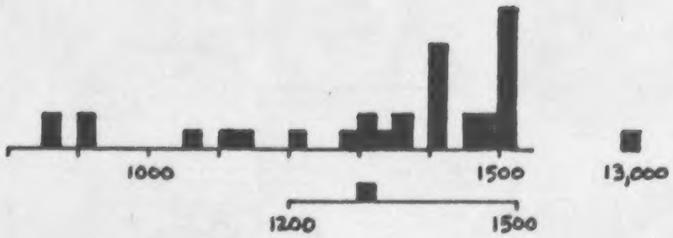
COMP 8



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 9

$$8127\,\overline{)474{,}257}$$

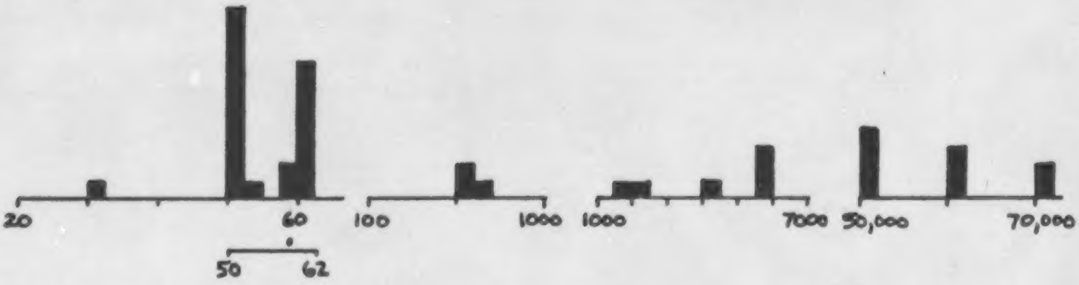
Acceptable Interval 50-62

Screening Data:

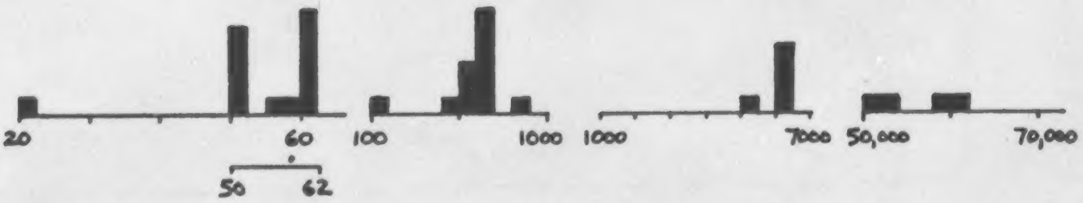
Time allowed: 17 sec. Time allowed: 15 sec.

	Grade 7-8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	21	12	17	21	19	20	21	25	22	30	31	30
Discrimination Index	.41	.41	.42	.26	.27	.26	.30	.34	.30	.50	.38	.41

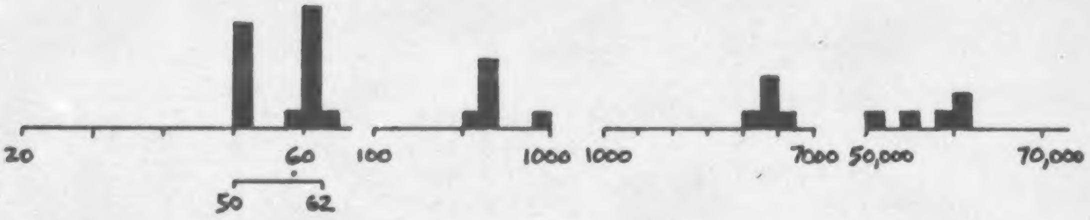
COMP 9



Grades 7,8
n=45 nr= 3



Grades 9,10
n=36 nr= 1



Grades 11,12
n=33 nr= 2

Note: ■ represents one response

Exercise 10

6346 ÷ 15

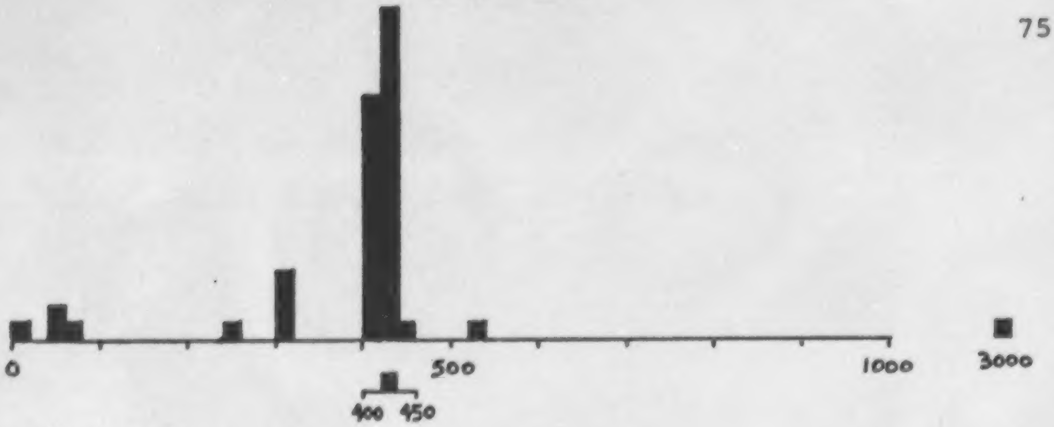
Acceptable Interval 400-450

Screening Data:

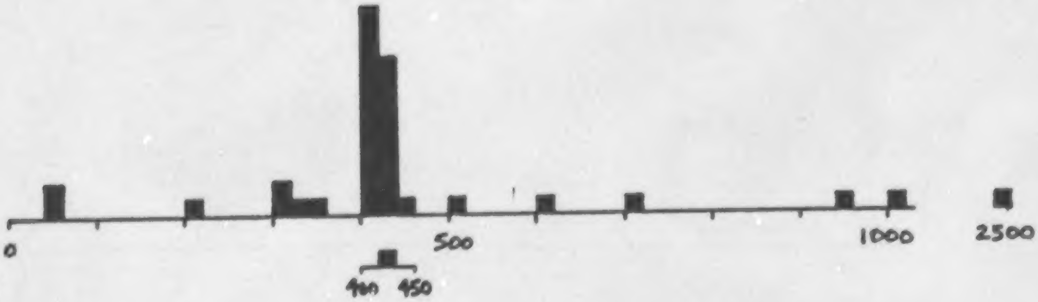
Time allowed: 17 sec. Time allowed: 15 sec.

	Grade 7 -8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	29	27	28	40	33	36	49	48	49	75	39	58
Discrimination Index	.46	.41	.44	.46	.32	.39	.46	.24	.37	.54	.38	.53

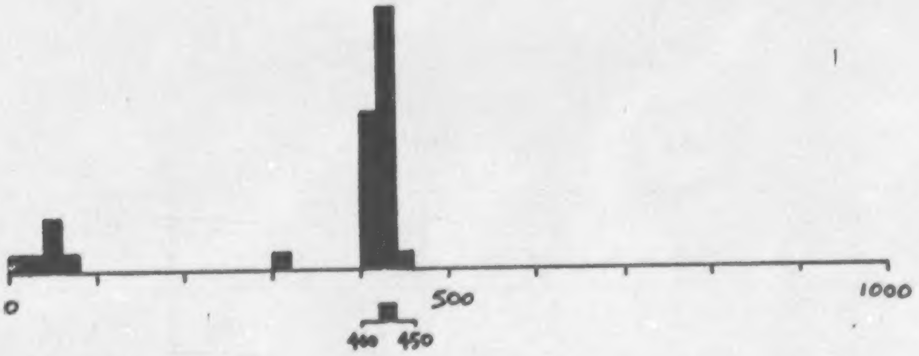
COMP 10



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise *11*

$$73\overline{)22}$$

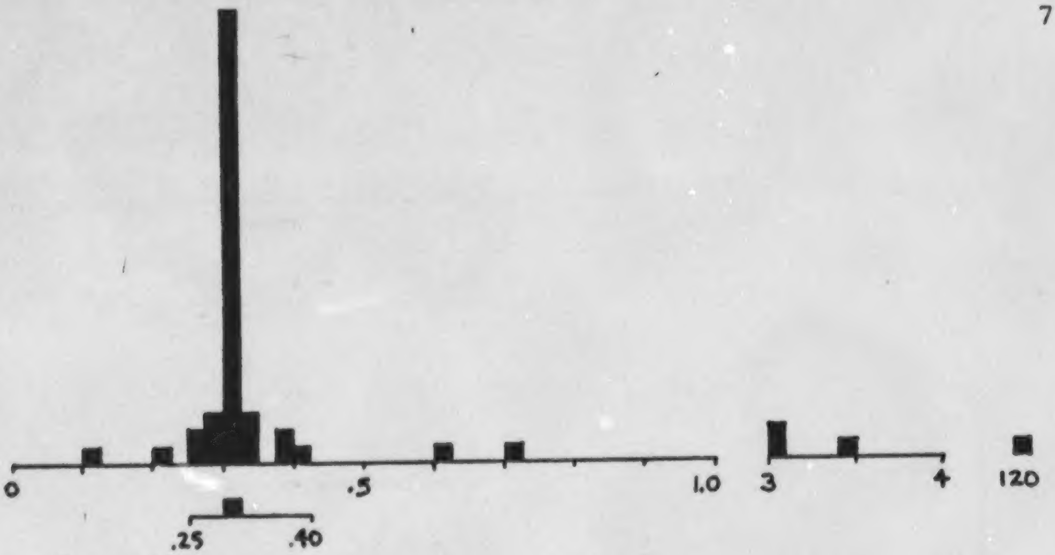
Acceptable Interval *.25 - .40*

Screening Data:

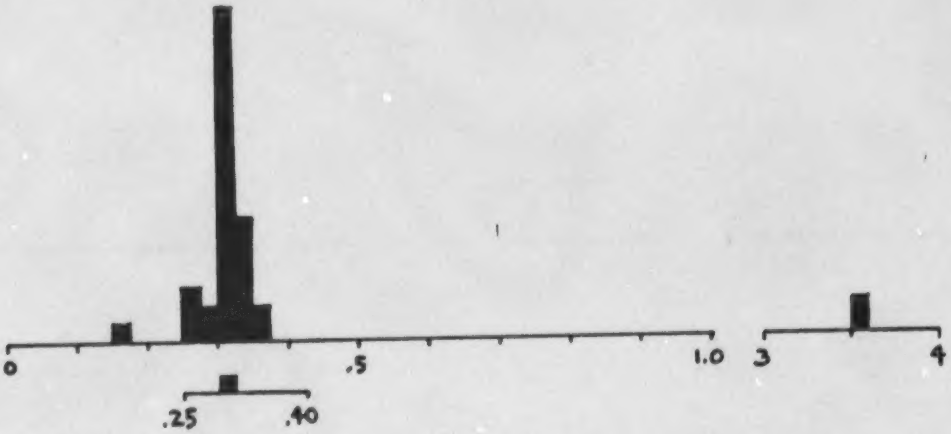
Time allowed: *12 sec.* Time allowed: *10 sec.*

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	43	33	38	65	58	61	62	60	62	79	65	73
Discrimination Index	.53	.55	.55	.55	.38	.46	.42	.26	.35	.65	.48	.58

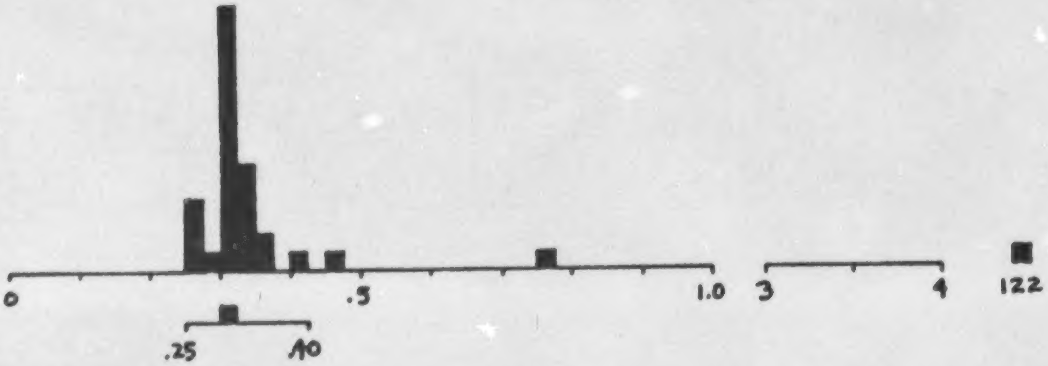
COMP11



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise *12*

$$200\overline{)800}$$

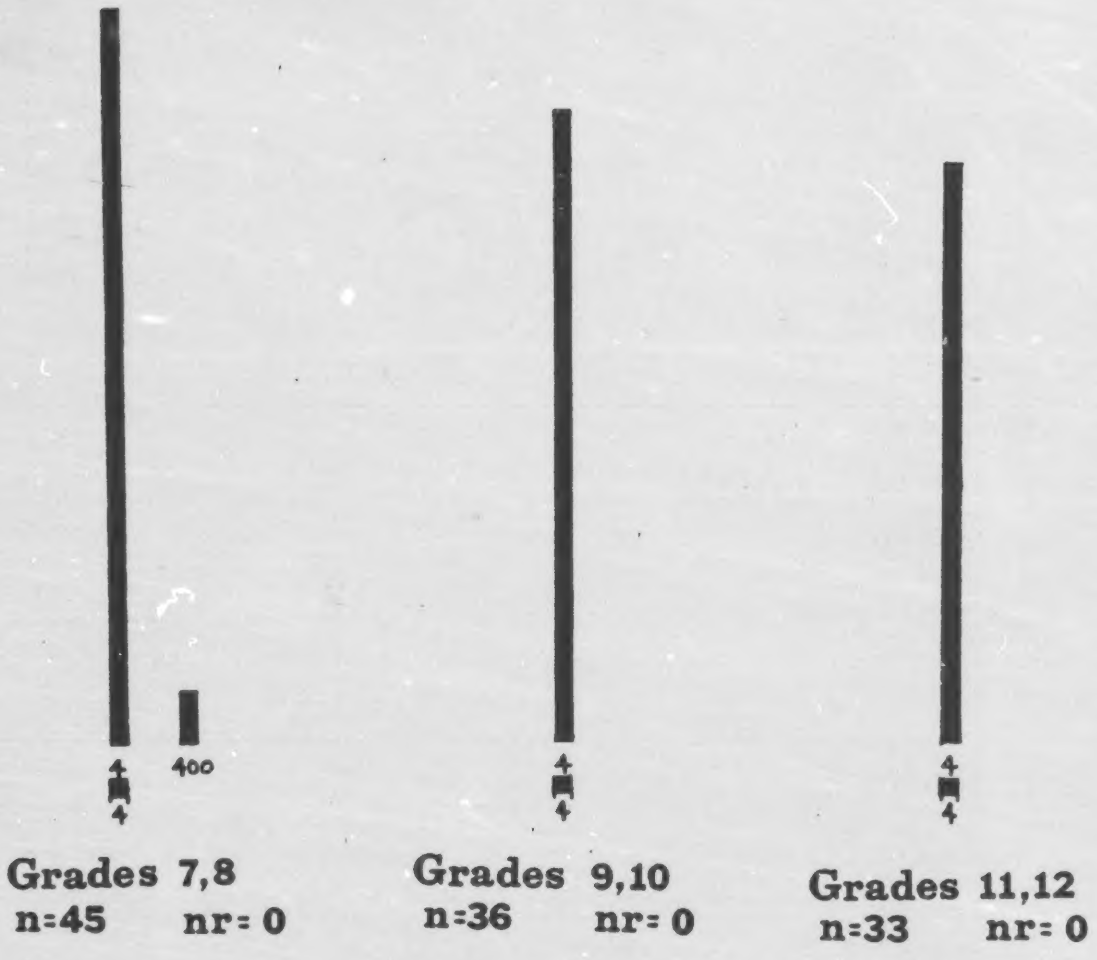
Acceptable Interval *4*

Screening Data:

Time allowed: *12 sec.*　　Time allowed: *10 sec.*

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	80	73	77	95	91	93	94	92	93	95	84	90
Discrimination Index	.38	.32	.36	.23	.31	.28	.26	.23	.25	.31	.33	.35

COMP 12



Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 13

$$8\overline{)713}$$

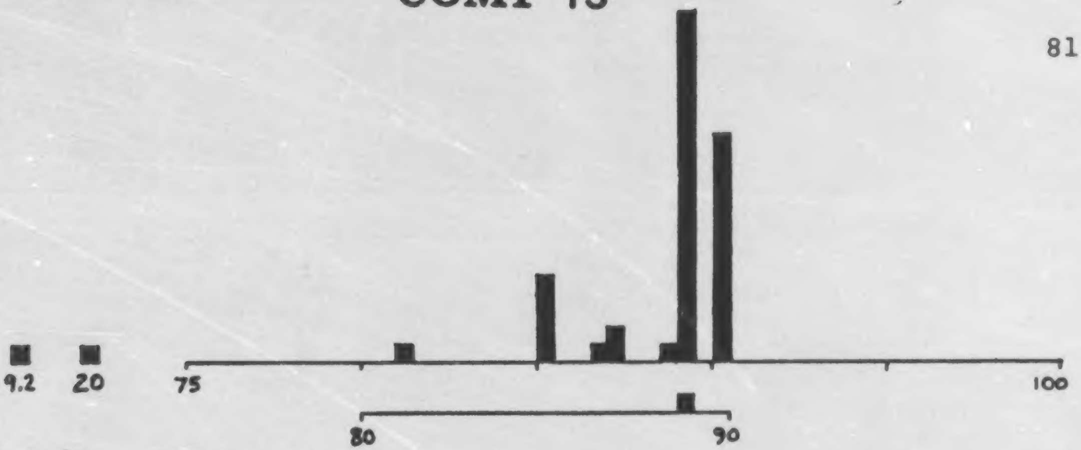
Acceptable Interval 80-90

Screening Data:

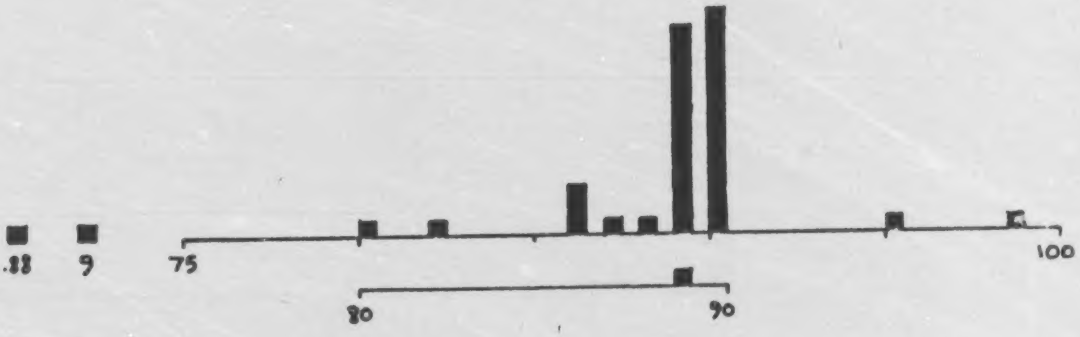
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7 - 8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	68	59	64	74	75	75	73	75	74	81	73	77
Discrimination Index	.48	.50	.50	.37	.28	.31	.31	.30	.30	.44	.22	.34

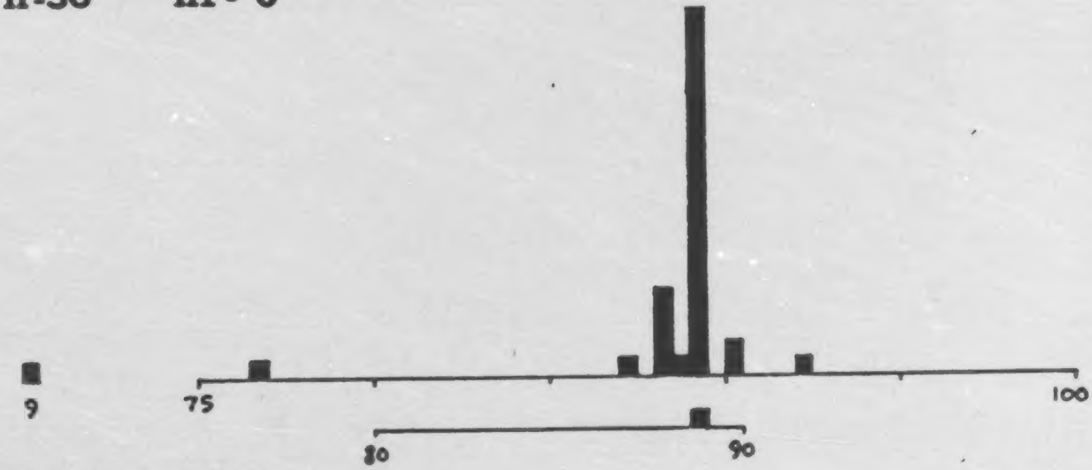
COMP 13



Grades 7,8
n=45 nr= 0



Grades 9,10
n=36 nr= 0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 14

6809 × 91

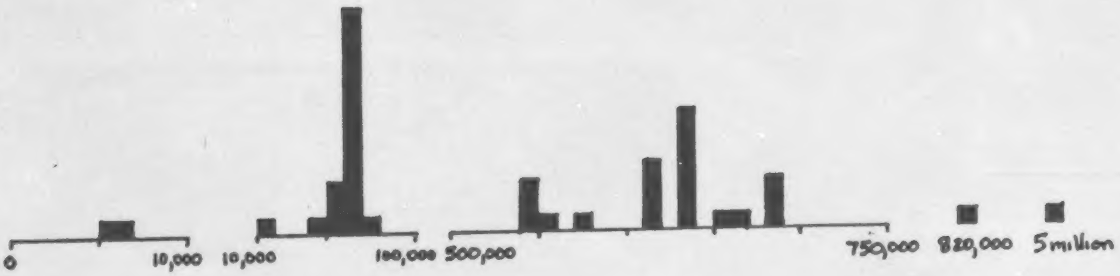
Acceptable Interval 610,000–700,000

Screening Data:

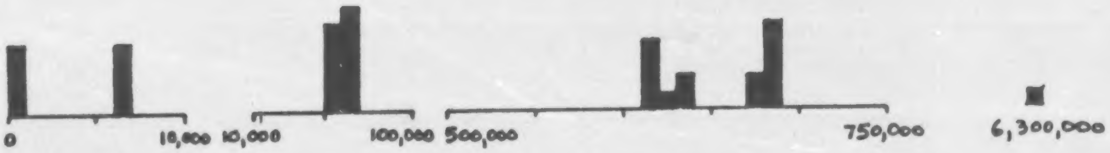
Time allowed: 12 sec.   Time allowed: 10 sec.

	Grade 7-8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	14	15	15	19	14	16	19	15	17	32	14	24
Discrimination Index	.35	.39	.36	.28	.34	.32	.33	.26	.30	.54	.25	.47

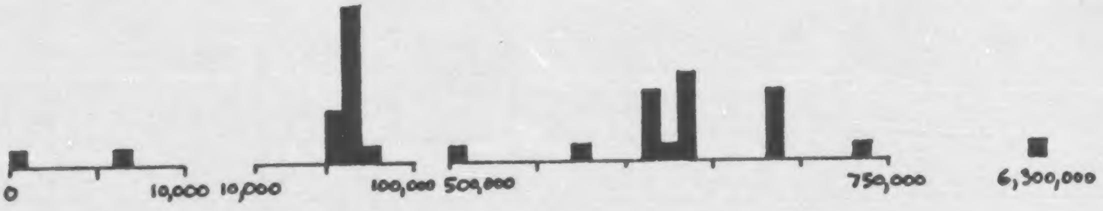
COMP 14



Grades 7,8
n=45 nr= 1



Grades 9,10
n=36 nr= 2



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 15

31 × 68 × 296

Acceptable Interval 600,000–634,000

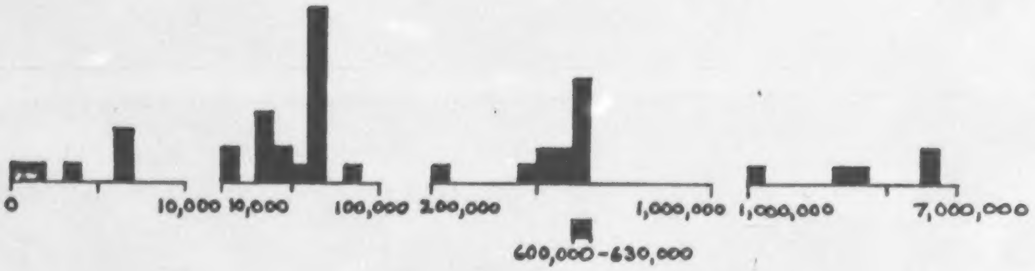
Screening Data:

Time allowed: 17 sec.

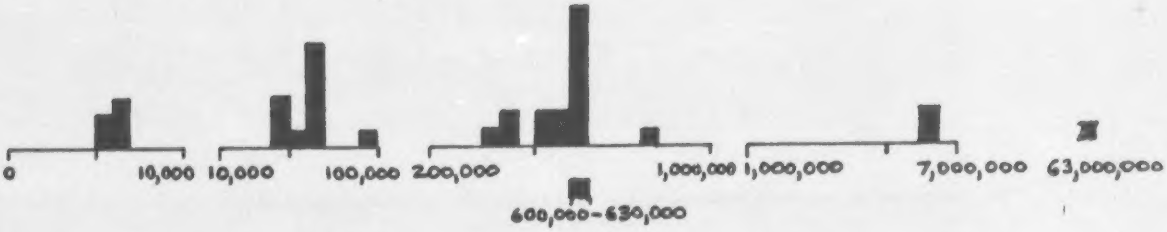
Time allowed: 15 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (206)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	3	2	3	6	8	7	7	6	7	25	12	19
Discrimination Index	.25	.16	.21	.33	.28	.29	.39	.29	.35	.54	.55	.55

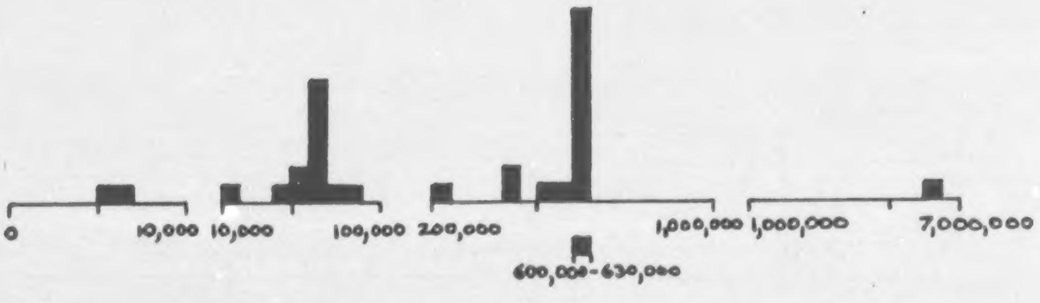
COMP 15



Grades 7,8
n=45 nr=2



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise **16**

$$\frac{347 \times 6}{43}$$

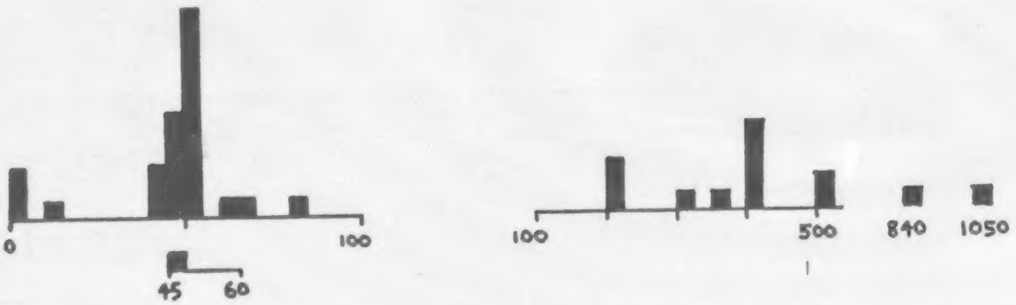
Acceptable Interval 45-60

Screening Data:

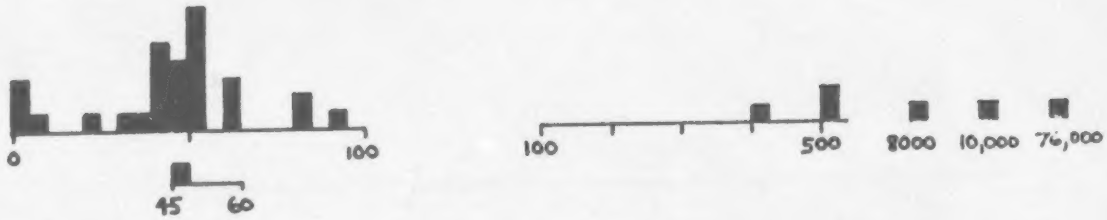
Time allowed: 11 sec. Time allowed: 15 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	10	6	8	22	17	19	30	25	28	33	29	31
Discrimination Index	.44	.32	.39	.25	.34	.30	.34	.40	.37	.50	.21	.37

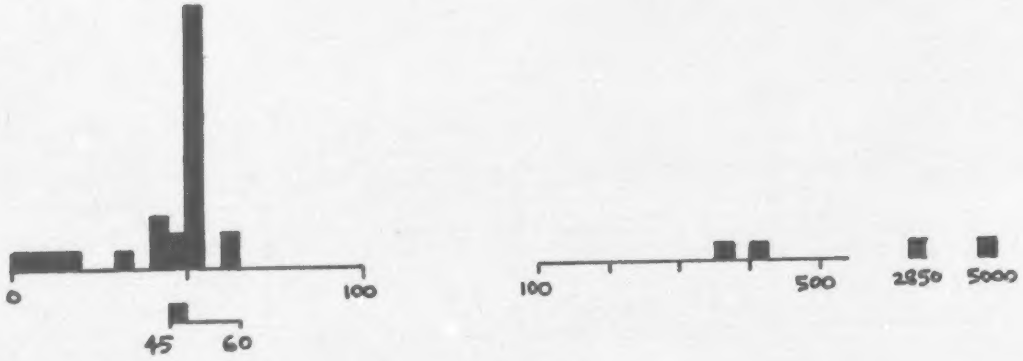
COMP 16



Grades 7,8
n=45 nr=3



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=2

Note: ■ represents one response

Exercise 17

308 - 2.85

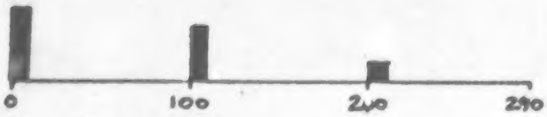
Acceptable Interval 300-306

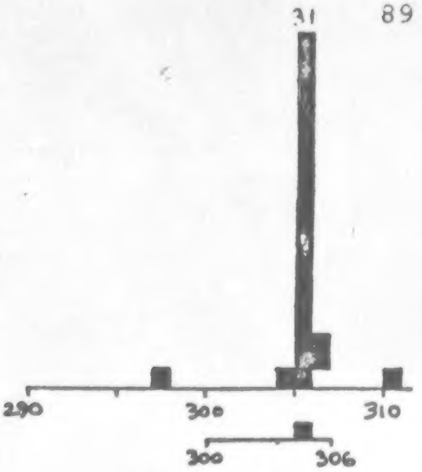
Screening Data:

Time allowed: 12 sec. Time allowed: 10 sec.

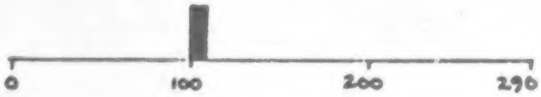
	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	39	30	35	67	56	60	64	64	64	86	49	69
Discrimination Index	.51	.60	.55	.26	.42	.36	.35	.44	.38	.36	.41	.46

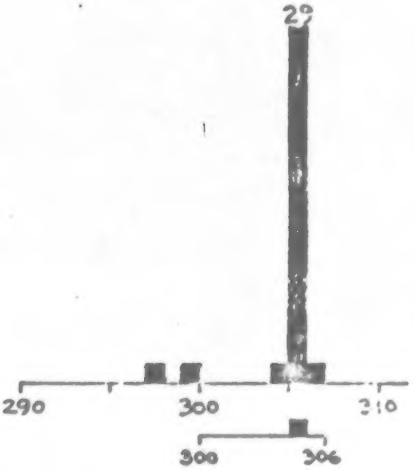
COMP17



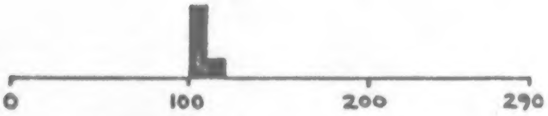


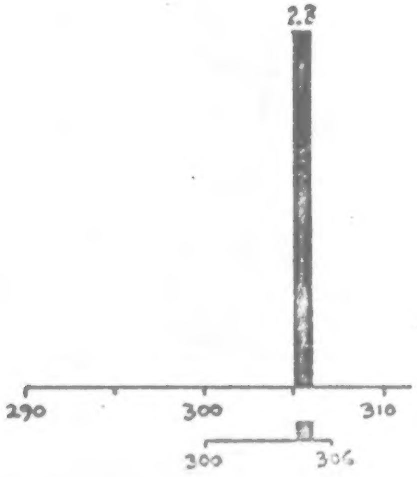
Grades 7,8
n=45 nr=1





Grades 9,10
n=36 nr=0



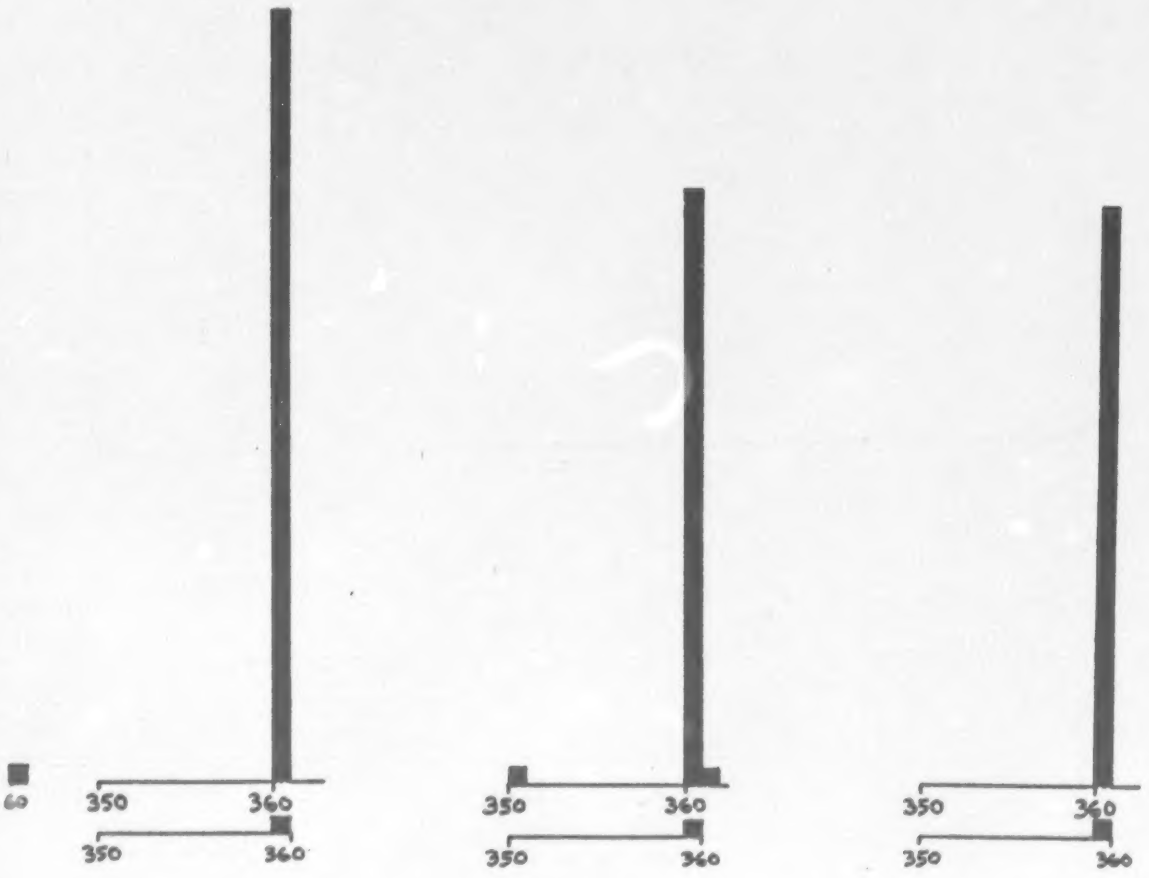


Grades 11,12
n=33 nr=0

Note: ▨ represents one response

PAGE 90 MISSING FROM DOCUMENT PRIOR TO ITS
BEING SHIPPED TO EDRS FOR FILMING.

COMP 18



Grades 7,8
n=45 nr=0

Grades 9,10
n=36 nr=0

Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 19

.7 + .002 + .81

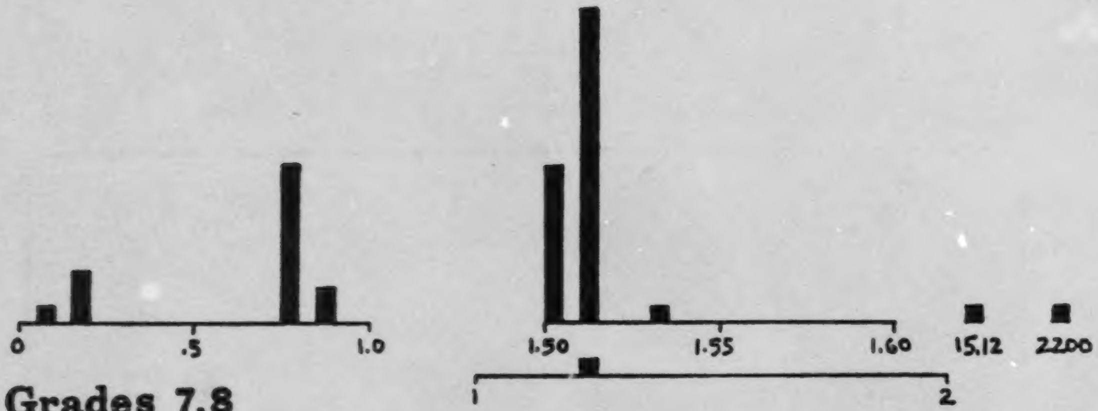
Acceptable Interval 1-2

Screening Data:

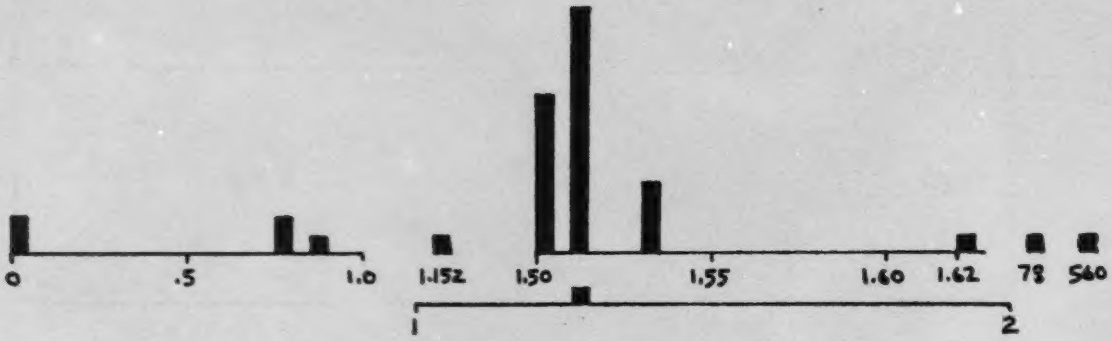
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	35	21	28	60	48	53	65	56	61	58	41	50
Discrimination Index	.55	.48	.53	.40	.42	.42	.40	.40	.40	.50	.67	.58

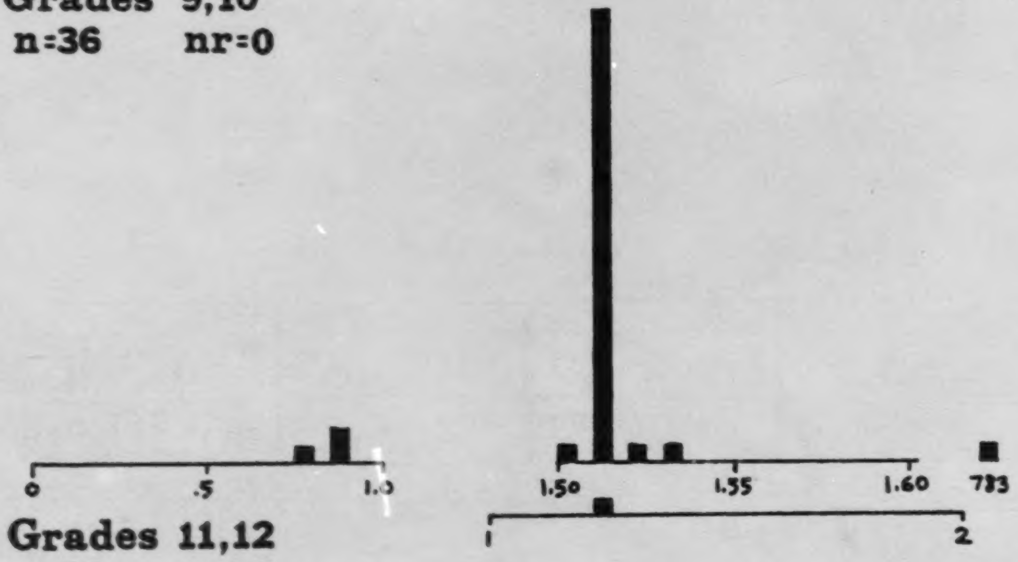
COMP 19



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 20

327 + 71.8

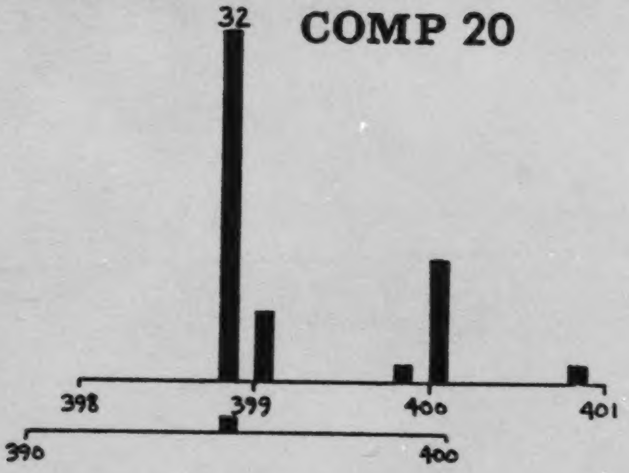
Acceptable Interval 390-400

Screening Data:

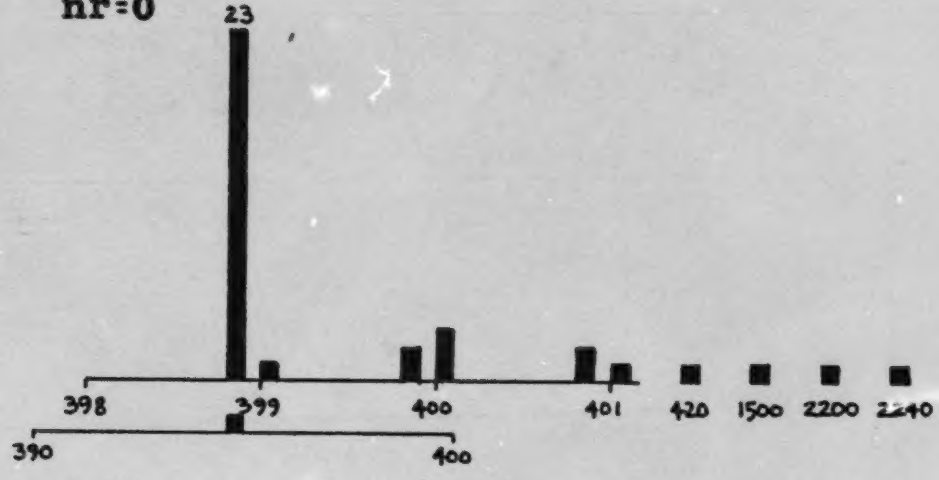
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	60	45	53	83	75	79	86	83	85	88	63	76
Discrimination Index	.61	.58	.61	.38	.24	.31	.29	.15	.23	.36	.50	.48

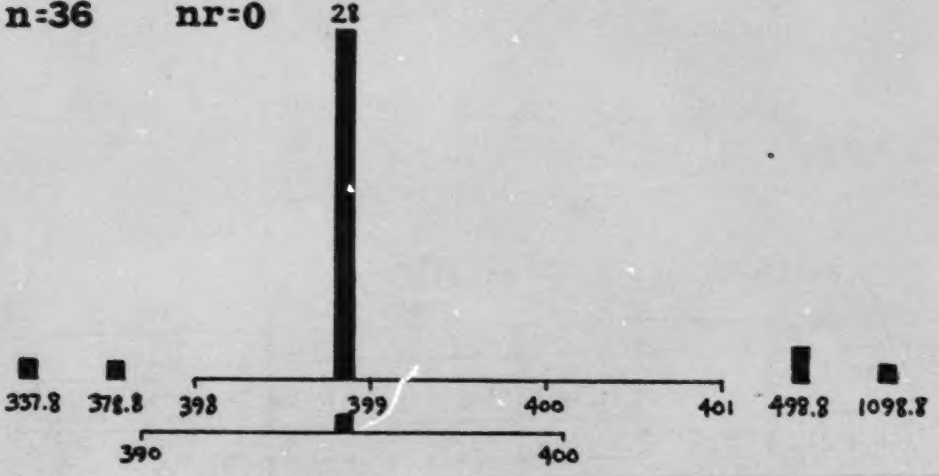
COMP 20



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 21

835.67 - .526

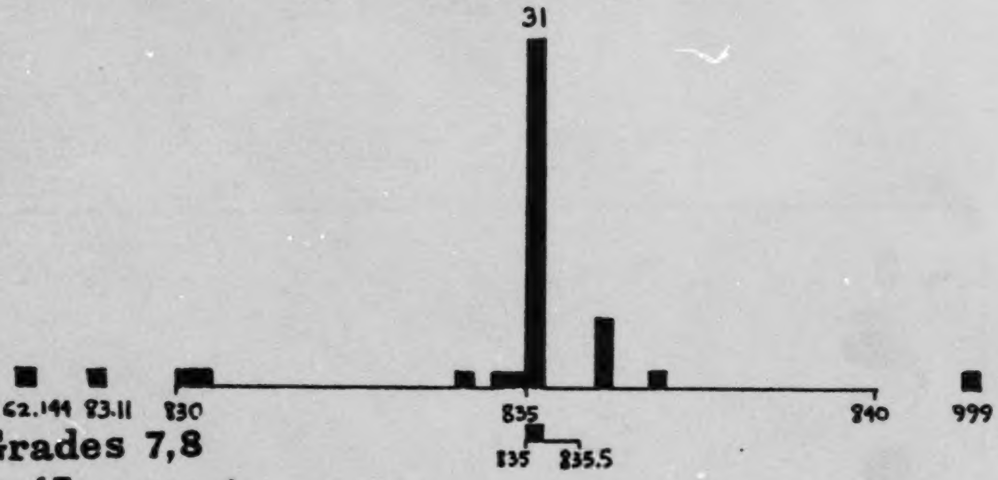
Acceptable Interval 835.0-835.5

Screening Data:

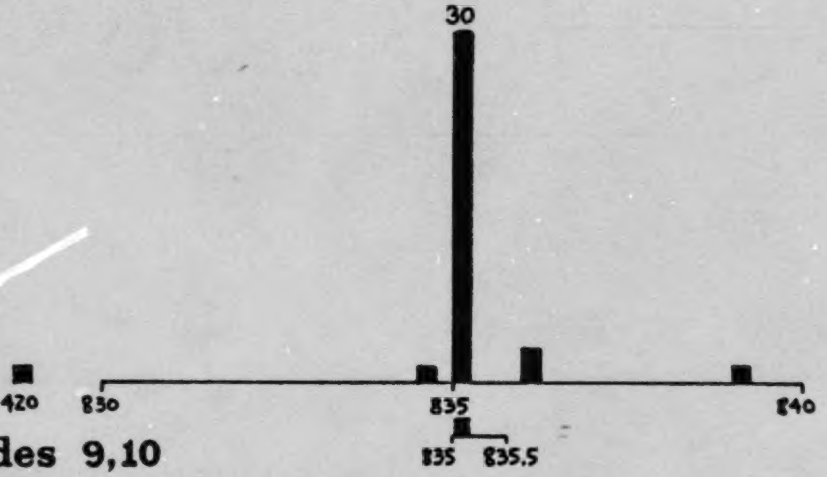
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7 -8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (206)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	37	20	29	69	46	56	62	48	57	53	53	53
Discrimination Index	.52	.35	.46	.36	.33	.38	.49	.33	.43	.50	.27	.37

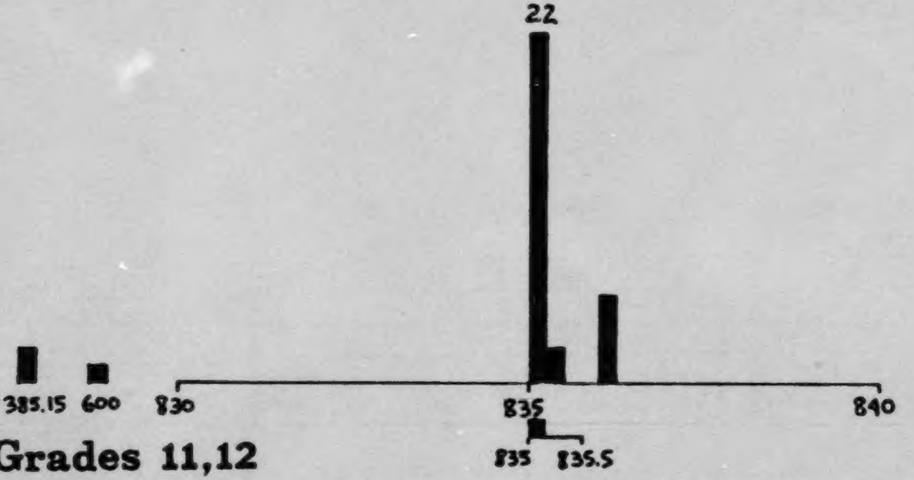
COMP 21



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 22

$$648 \div 1.06$$

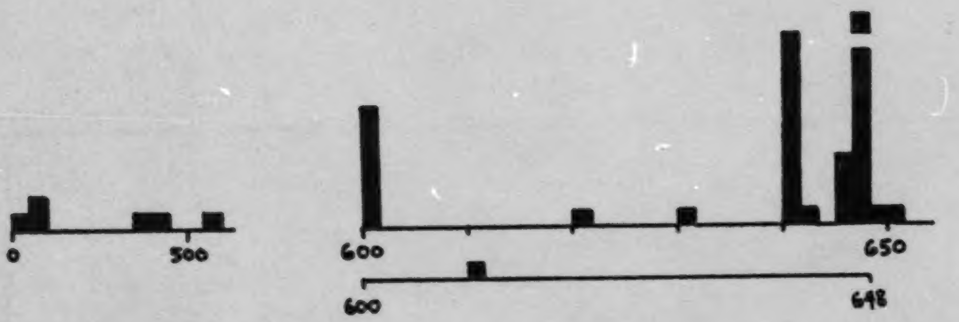
Acceptable Interval 600-648

Screening Data:

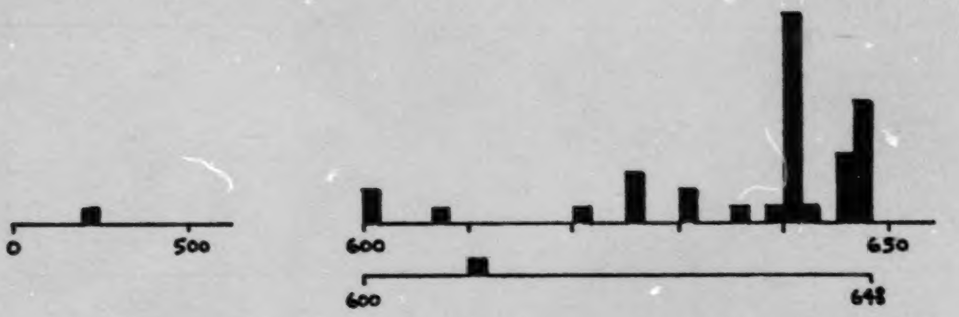
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	46	26	36	77	59	66	65	49	59	74	67	71
Discrimination Index	.63	.53	.60	.40	.34	.39	.43	.50	.47	.54	.53	.52

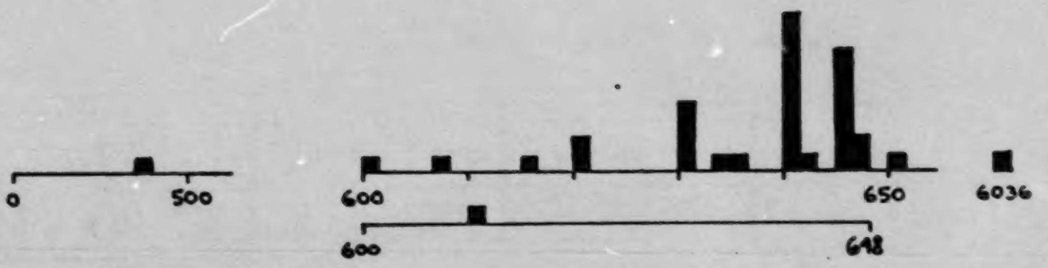
COMP 22



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 23

$1\frac{1}{2} \times 1.67$

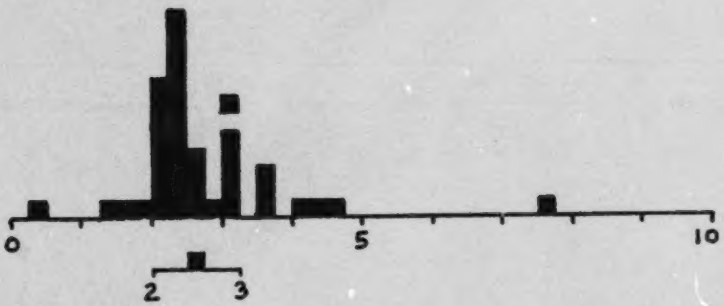
Acceptable Interval 2-3

Screening Data:

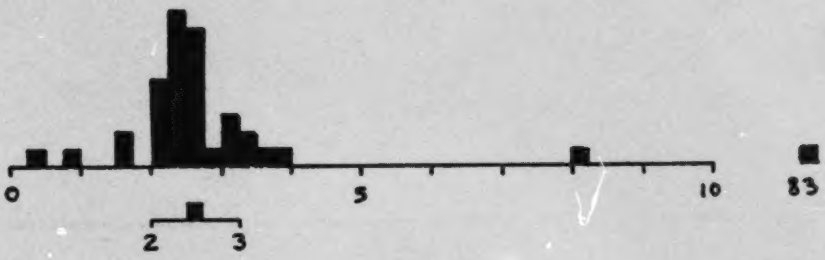
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	27	19	23	54	39	45	44	33	40	67	53	60
Discrimination Index	.48	.50	.49	.49	.48	.50	.37	.51	.43	.50	.40	.47

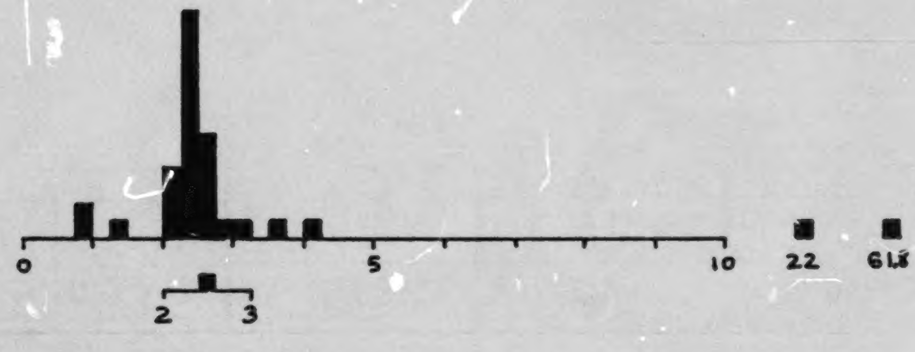
COMP 23



Grades 7,8
n=45 nr=3



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise 24

$$1\frac{7}{8} \times 1.19 \times 4$$

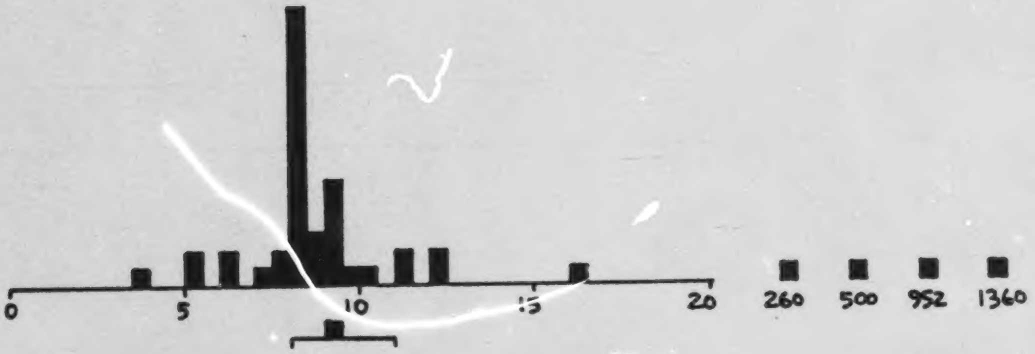
Acceptable Interval 8-10

Screening Data:

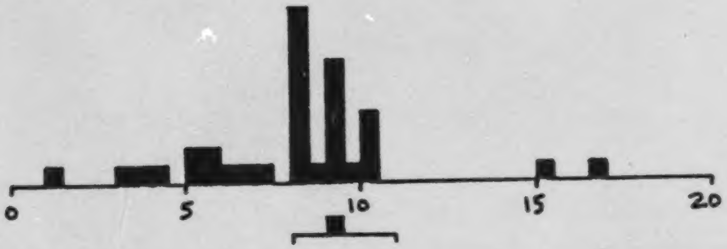
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	18	17	17	37	25	30	38	25	32	58	35	47
Discrimination Index	.46	.45	.45	.44	.38	.43	.37	.35	.37	.45	.54	.53

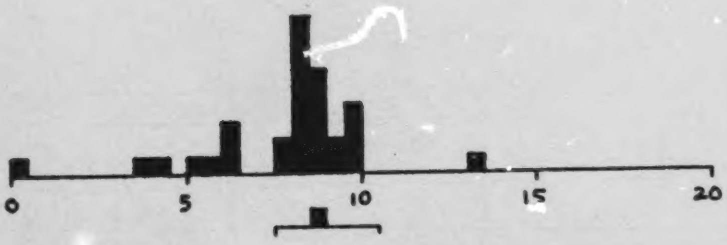
COMP 24



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 25

$$61.3 \times .8$$

Acceptable Interval 48.0-61.3

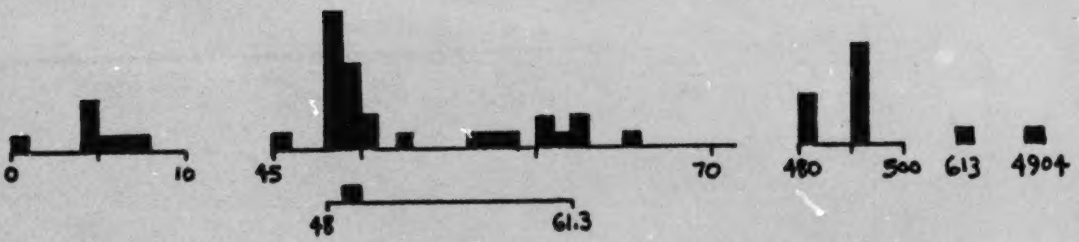
Screening Data:

Time allowed: 12 sec.

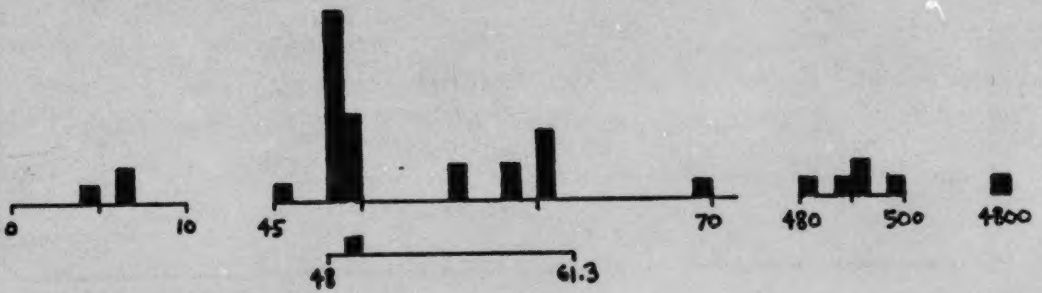
Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	37	44	41	49	49	49	48	55	51	61	41	52
Discrimination Index	.29	.28	.27	.40	.26	.31	.25	.20	.23	.37	.19	.34

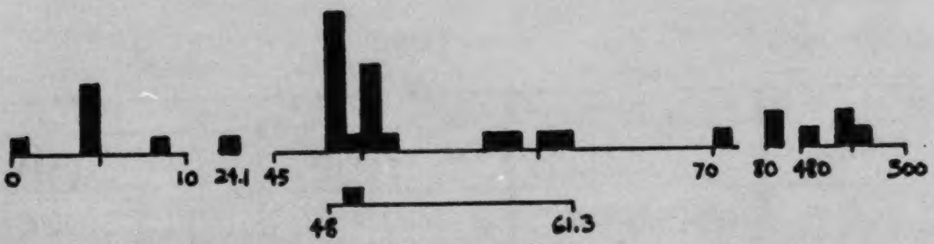
COMP 25



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise **26**

$$5.1 \times 4.8 \times 6.3$$

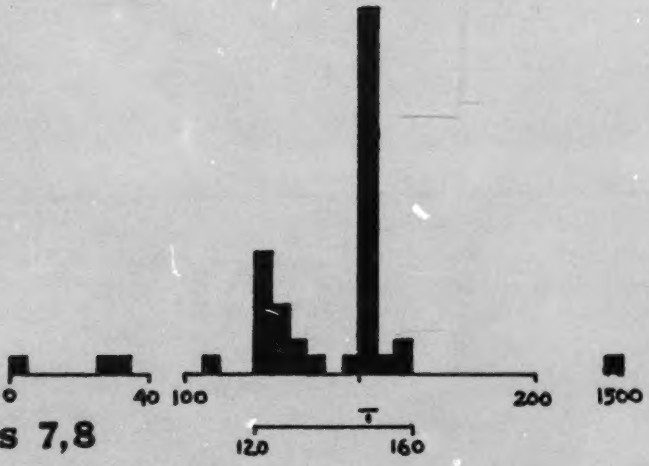
Acceptable Interval 120~160

Screening Data:

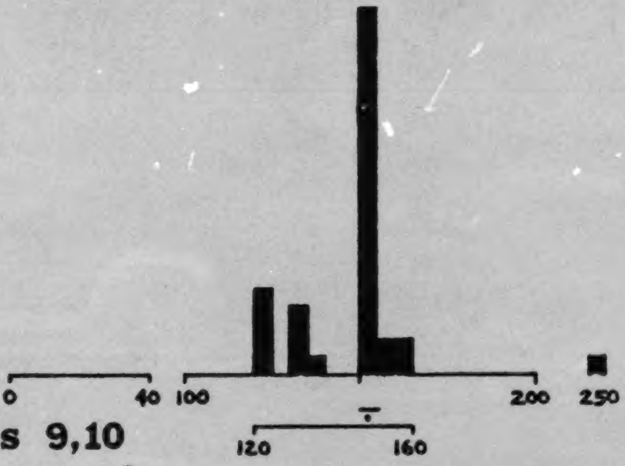
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	45	29	37	70	50	58	58	46	53	77	51	68
Discrimination Index	.61	.63	.63	.48	.48	.51	.44	.48	.46	.61	.43	.55

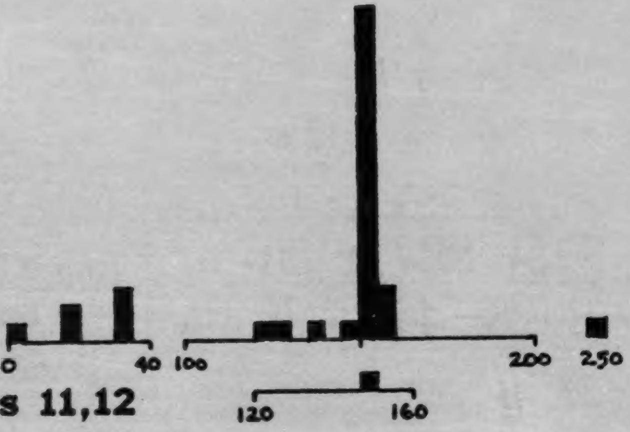
COMP 26



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 26

$$5.1 \times 4.8 \times 6.3$$

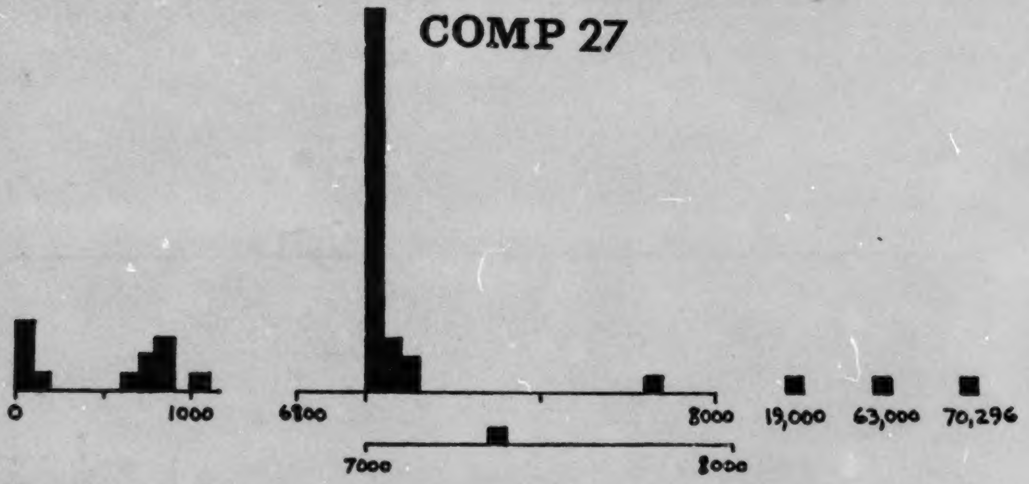
Acceptable Interval 120~160

Screening Data:

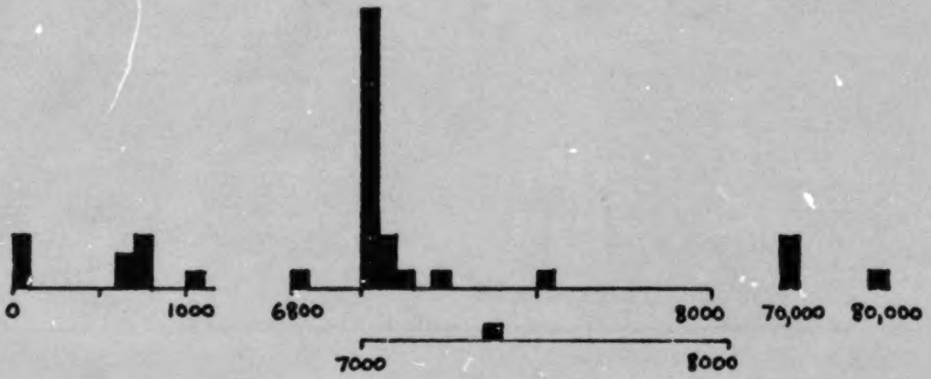
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	45	29	37	70	50	58	58	46	53	77	57	68
Discrimination Index	.61	.63	.63	.48	.48	.51	.44	.48	.46	.61	.43	.55

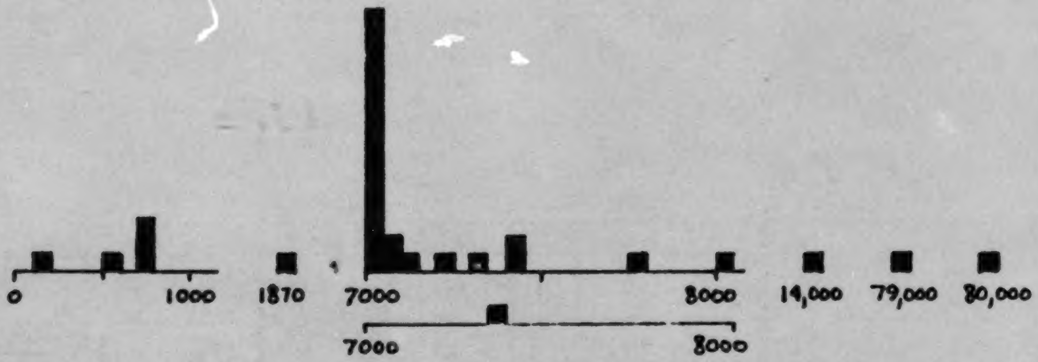
COMP 27



Grades 7,8
n=45 nr=2



Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 28

98.6 × .041

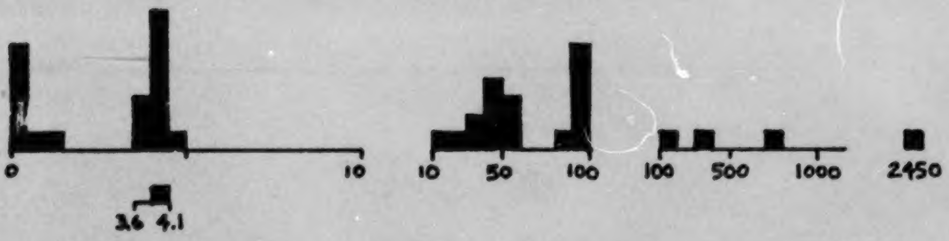
Acceptable Interval 3.6-41

Screening Data:

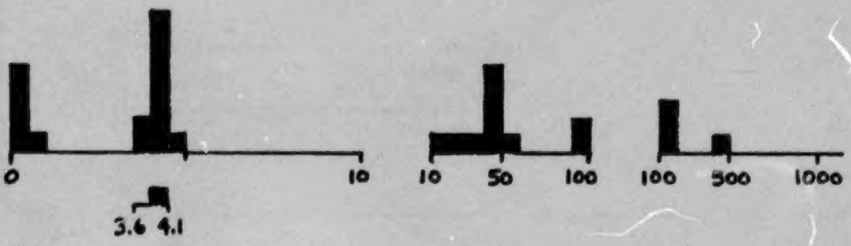
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	10	6	8	12	8	10	15	8	12	25	8	17
Discrimination Index	.28	.28	.29	.29	.15	.23	.39	.24	.34	.45	.38	46

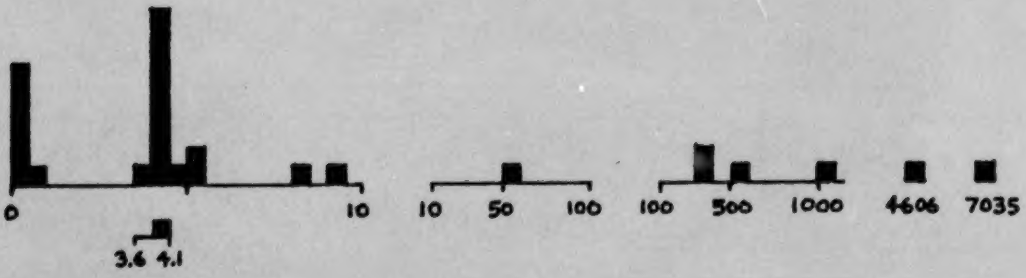
COMP 28



Grades 7,8
n=45 nr=3



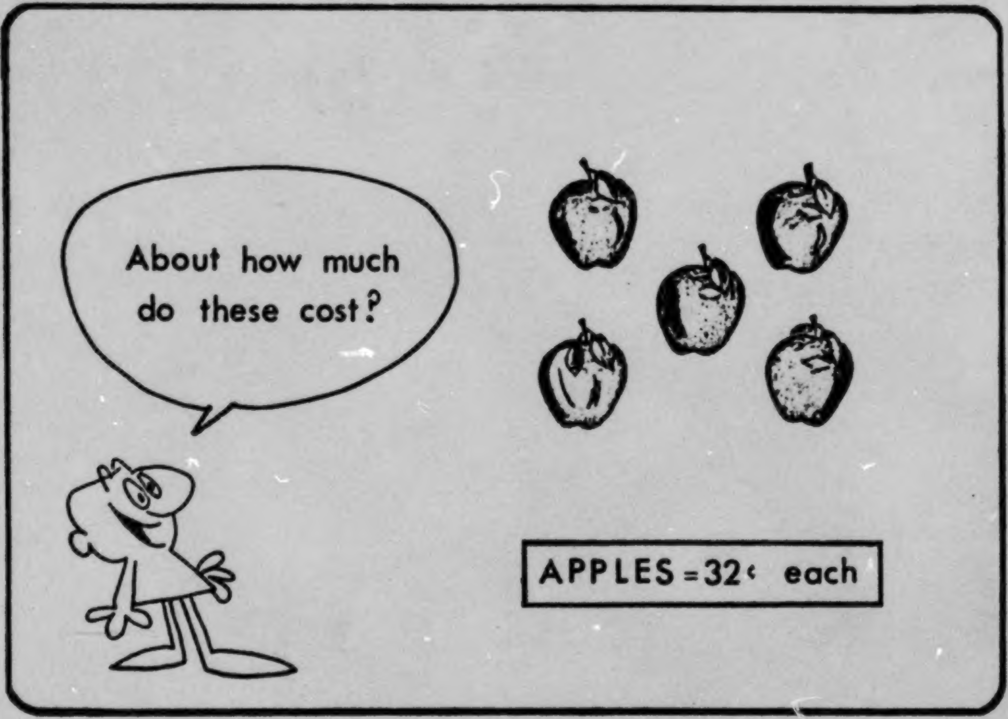
Grades 9,10
n=36 nr=4



Grades 11,12
n=33 nr=2

Note: ■ represents one response

Exercise 1


About how much do these cost?
APPLES = 32¢ each

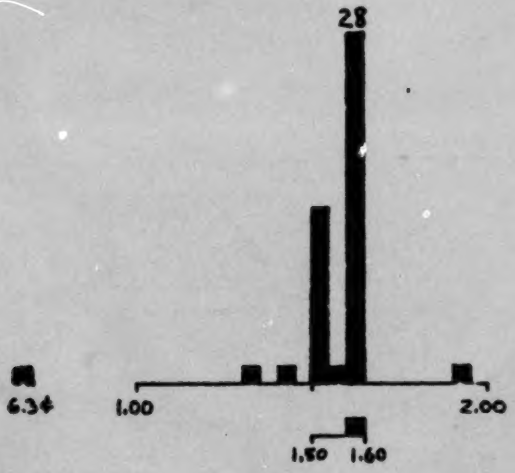
Acceptable Interval 1.50-1.60

Screening Data:

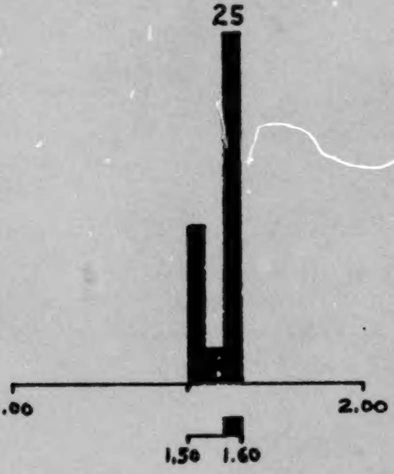
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	74	74	74	86	81	81	84	83	84	86	80	83
Discrimination Index	.34	.23	.28	.42	.24	.31	.37	.14	.28	.53	.45	.49

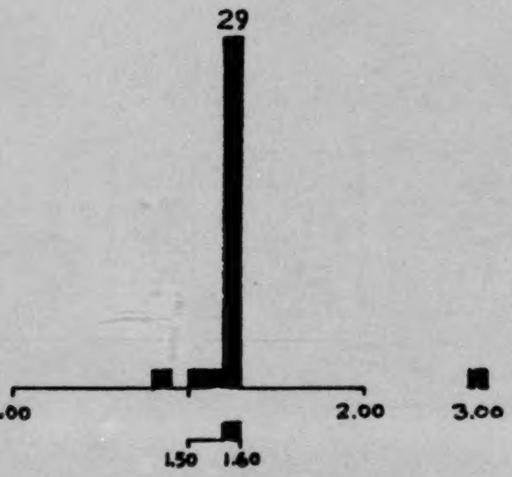
APPL 1



Grades 7,8
n=45 nr=2



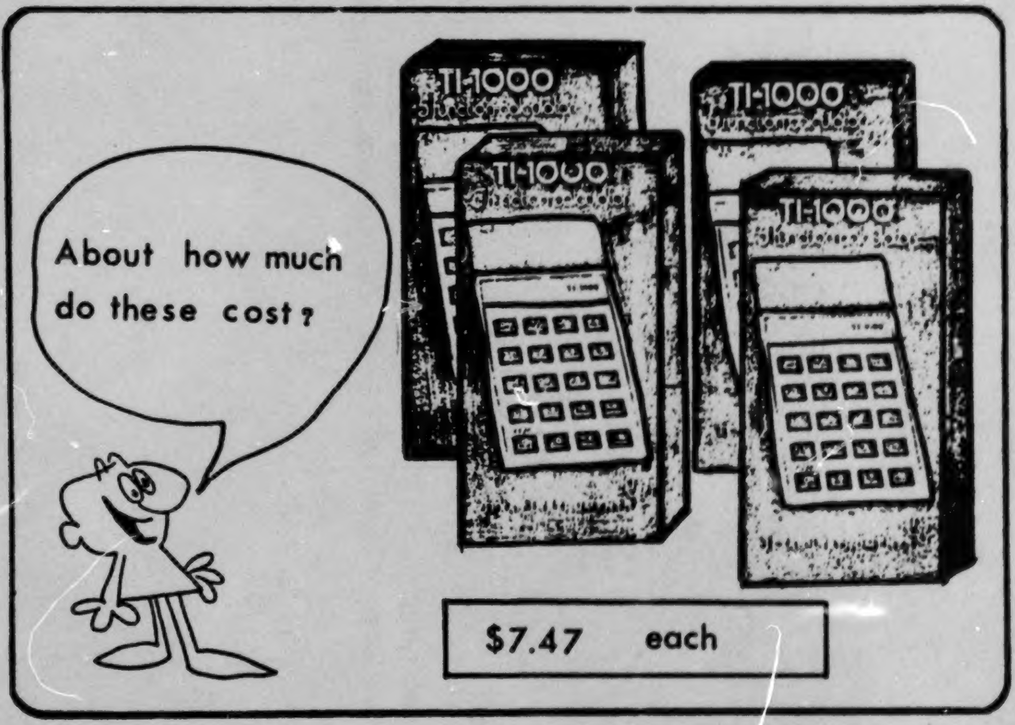
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 2


About how much do these cost?
TI-1000
TI-1000
TI-1000
TI-1000
$7.47 each

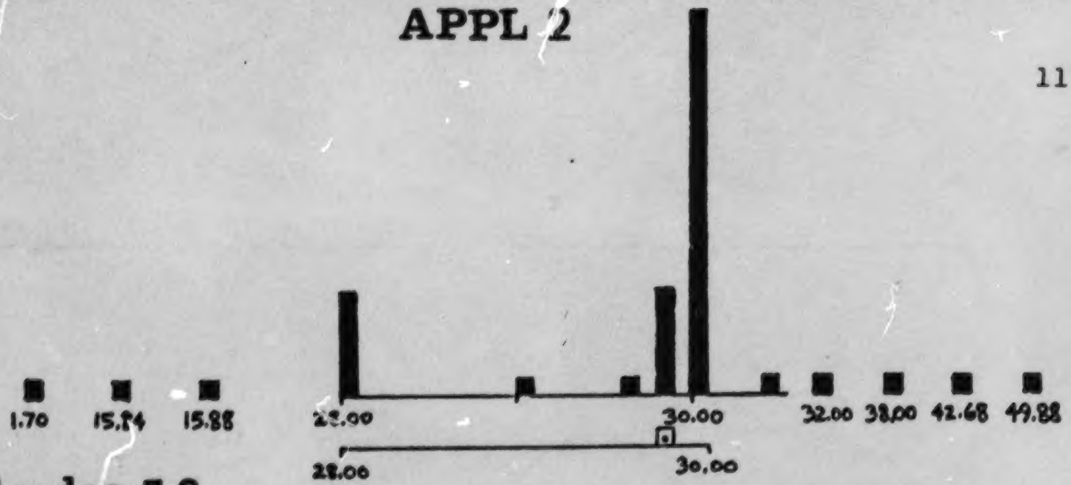
Acceptable Interval 28~30

Screening Data:

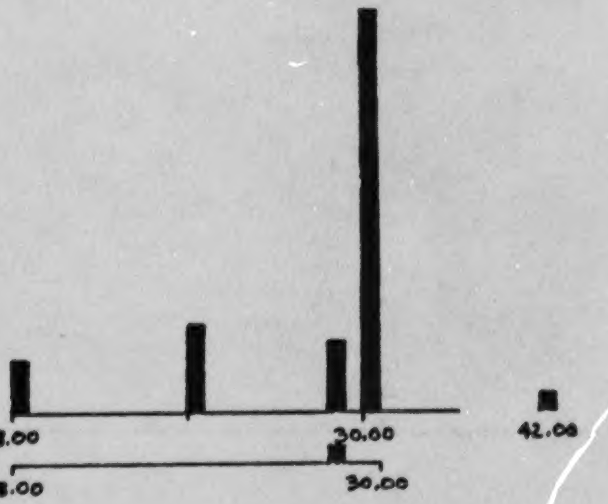
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M ([illegible]7)	F ([illegible])	T (106)
Percent Correct	64	54	59	69	76	73	72	72	72	72	67	70
Discrimination Index	.48	.33	.42	.34	.21	.25	.40	.26	.34	.37	.37	.37

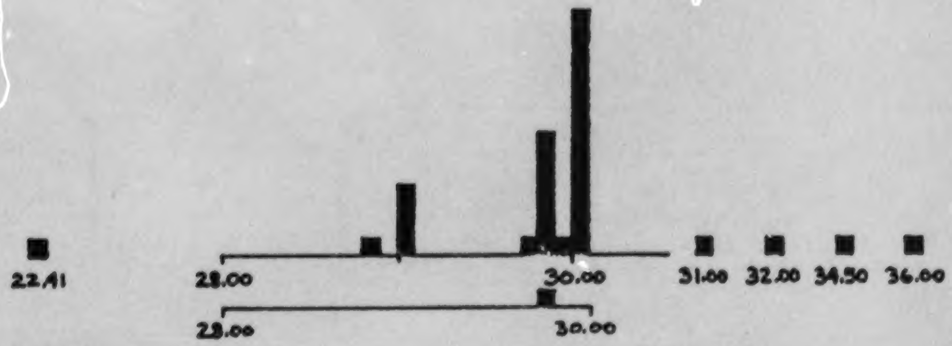
APPL 2



Grades 7,8
n=45 nr=0



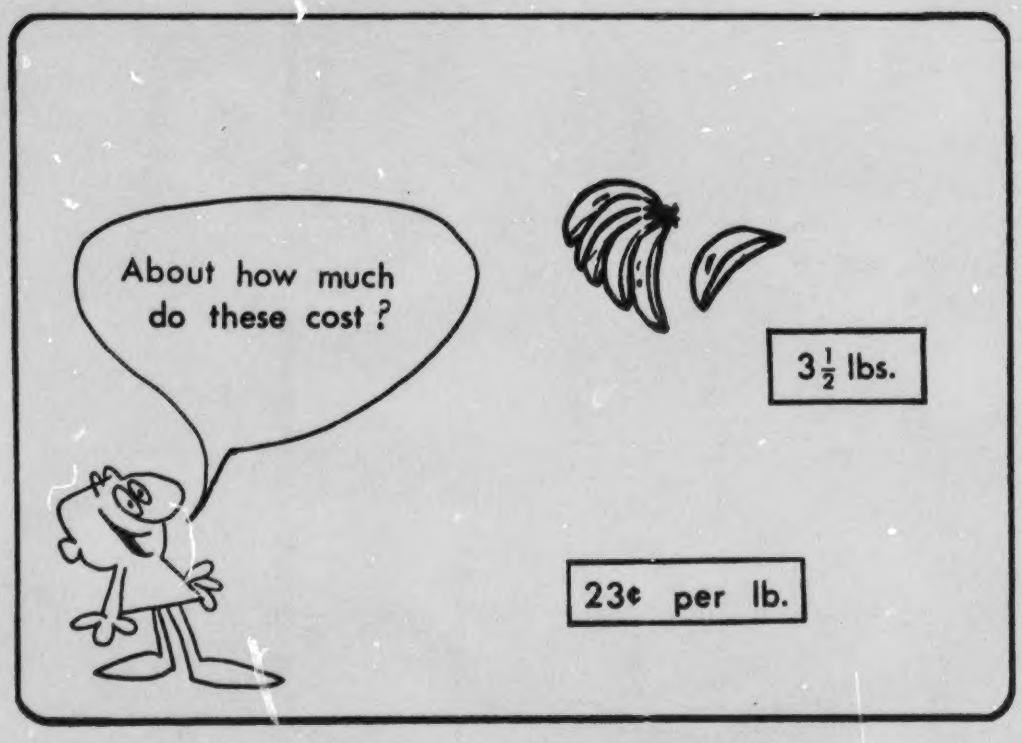
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 3


About how much do these cost?
3½ lbs.
23¢ per lb.

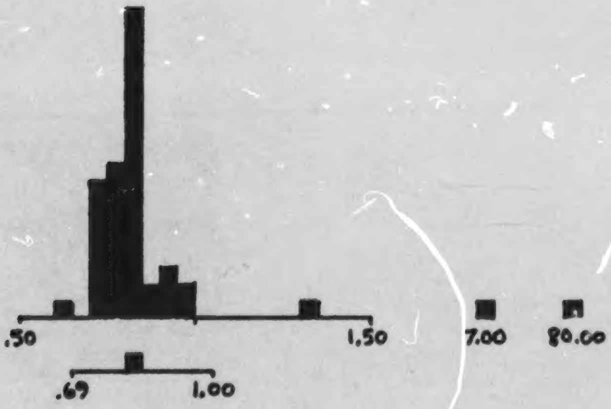
Acceptable Interval .69-1.00

Screening Data:

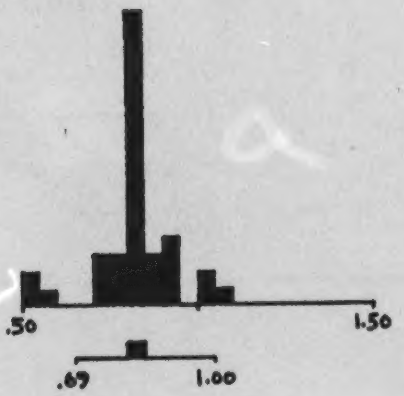
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M(222)	F(209)	T(431)	M(154)	F(205)	T(359)	M(165)	F(126)	T(291)	M(57)	F(49)	T(106)
Percent Correct	64	46	55	79	67	72	78	67	73	88	78	83
Discrimination Index	.48	.45	.48	.24	.40	.35	.34	.43	.38	.51	.36	.44

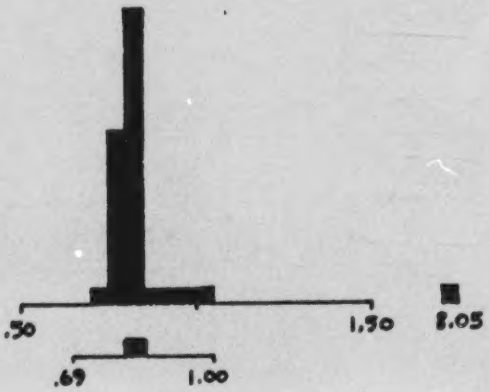
APPL 3



Grades 7,8
n=45 nr=0



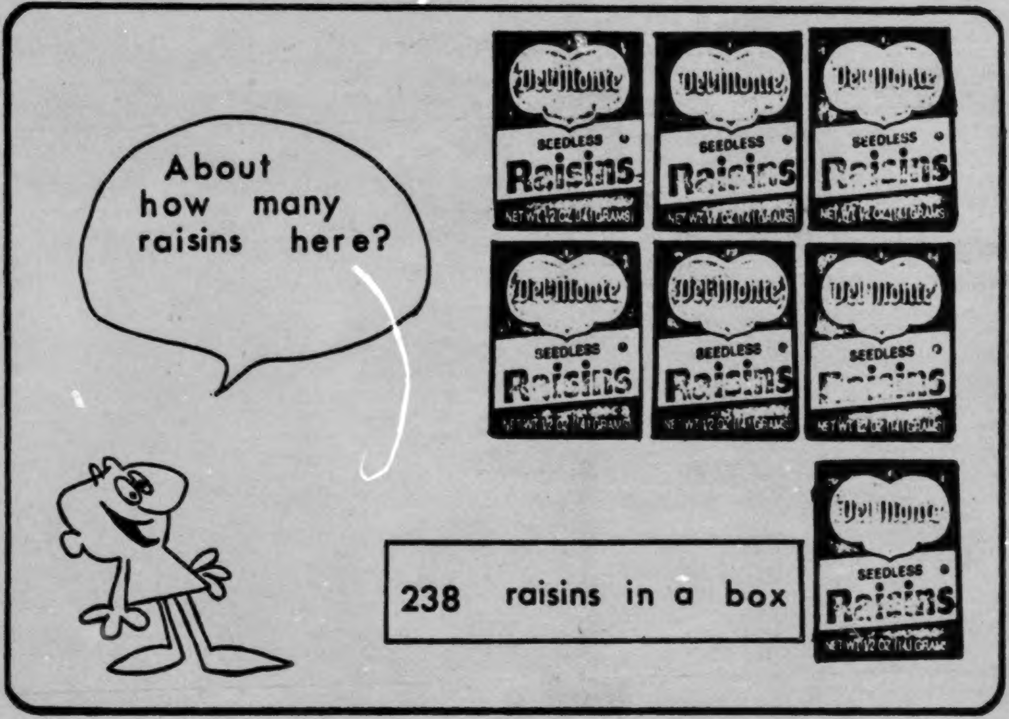
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 4


About how many raisins here?
238 raisins in a box
Del Monte SEEDLESS Raisins
NET WT 1/2 OZ (14.1 GRAMS)

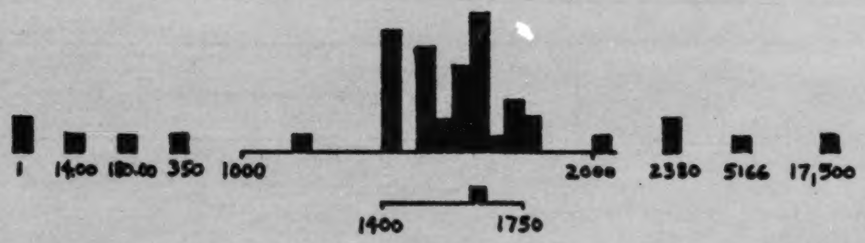
Acceptable Interval 1400-1750

Screening Data:

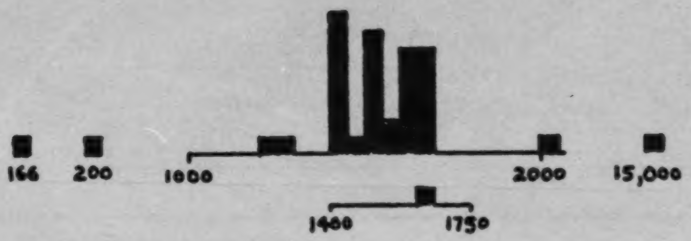
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T ([illegible])
Percent Correct	47	44	46	60	51	55	61	55	59	70	76	73
Discrimination Index	.41	.54	.46	.29	.43	.38	.39	.48	.43	.40	.40	.39

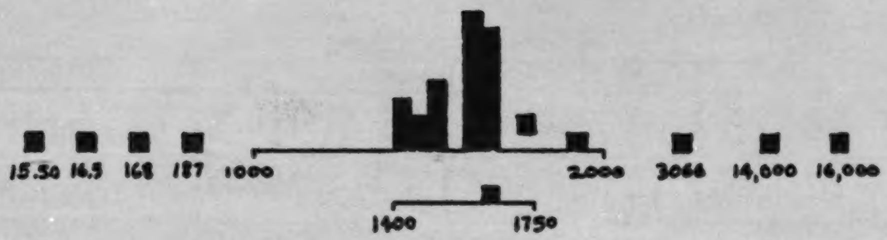
APPL 4



Grades 7,8
n=45 nr=0



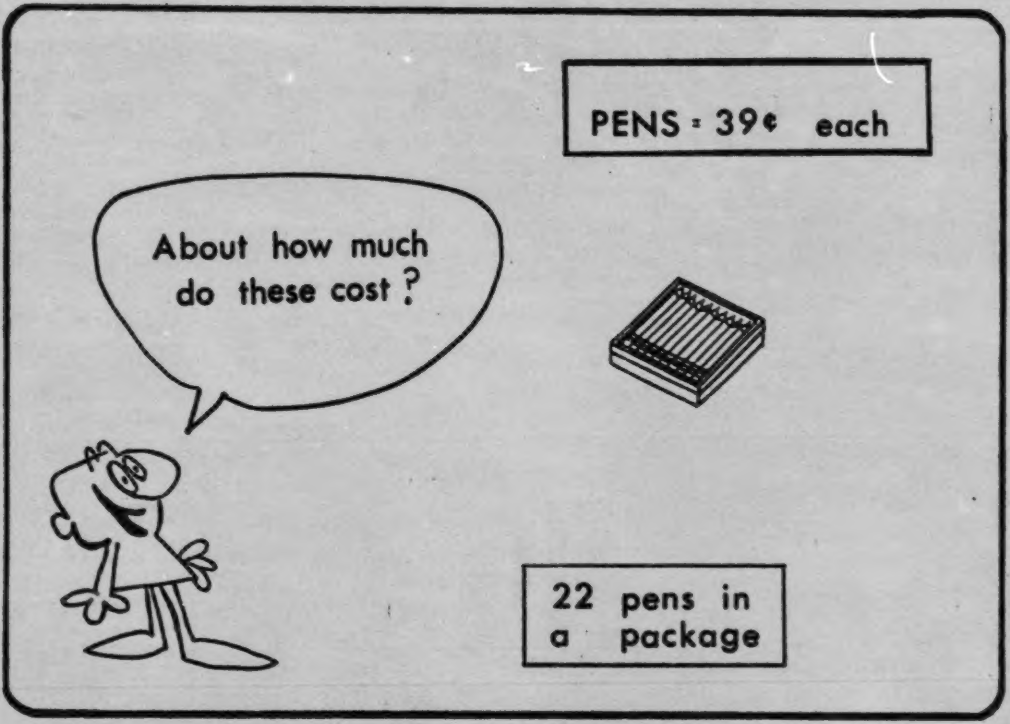
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise **5**


PENS = 39¢ each
About how much do these cost?
22 pens in a package

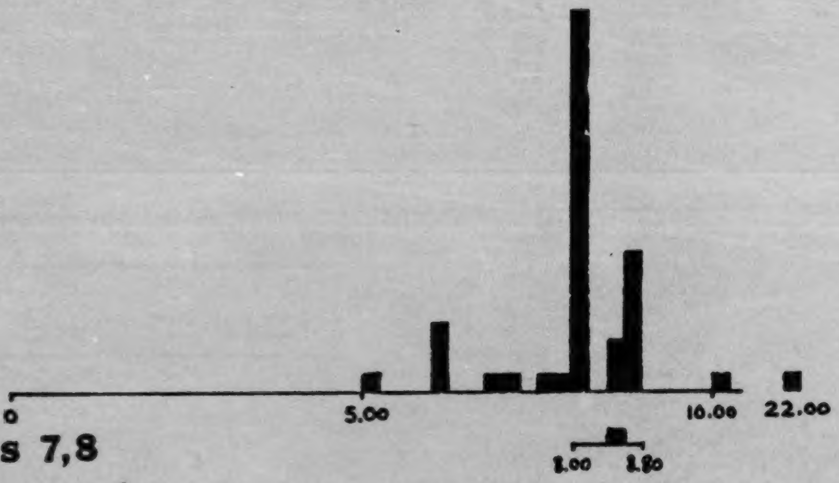
Acceptable Interval 8.00-8.80

Screening Data:

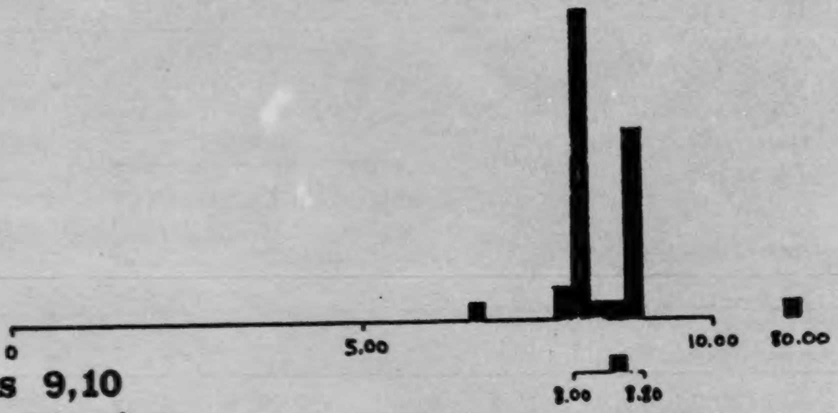
Time allowed: 12sec. Time allowed: 10sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	25	24	25	38	34	36	45	36	41	74	51	63
Discrimination Index	.36	.44	.39	.44	.45	.45	.53	.42	.50	.56	.37	.48

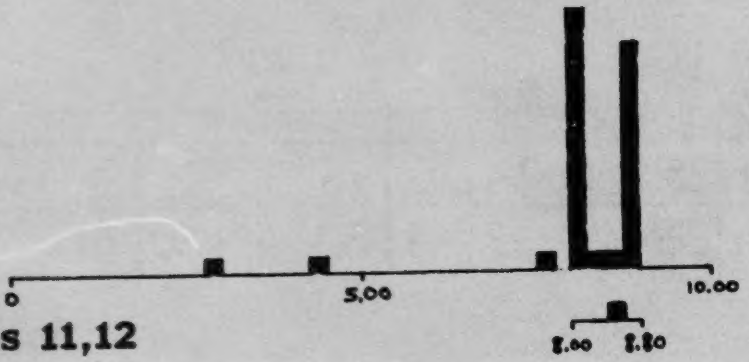
APPL 5



Grades 7,8
n=45 nr=1



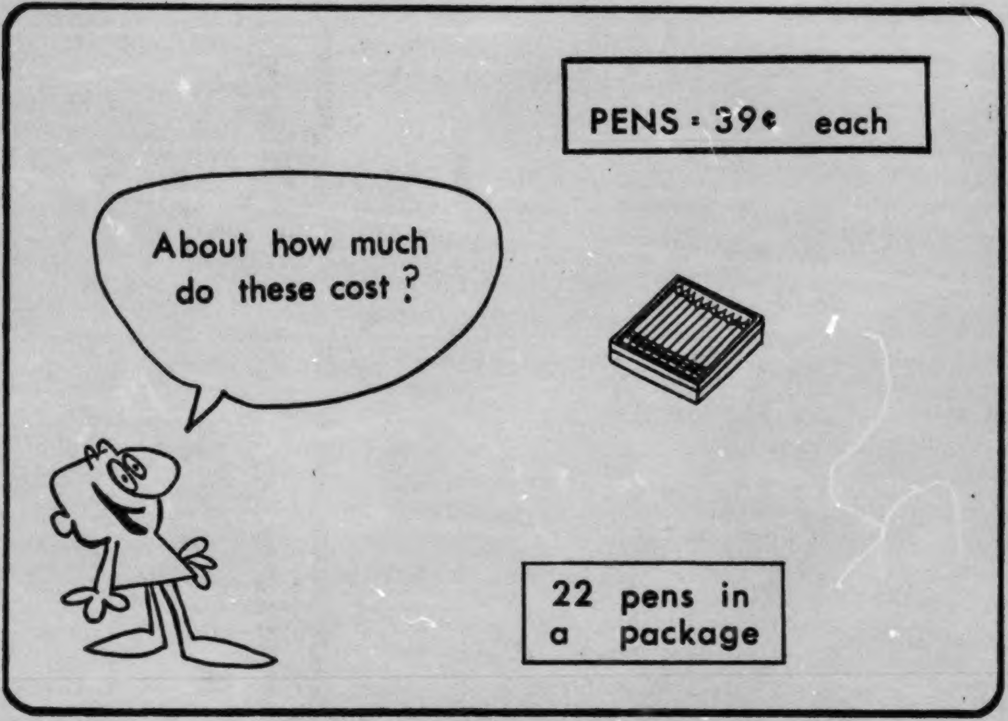
Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 5


PENS = 39¢ each
About how much do these cost?
22 pens in a package

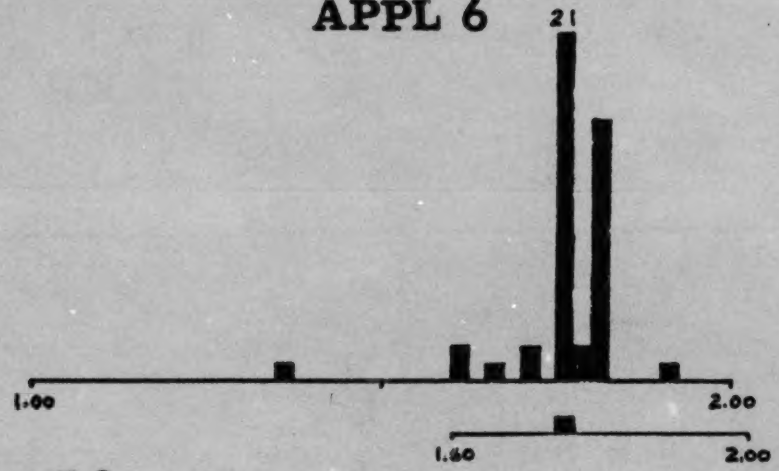
Acceptable Interval 8.00~8.80

Screening Data:

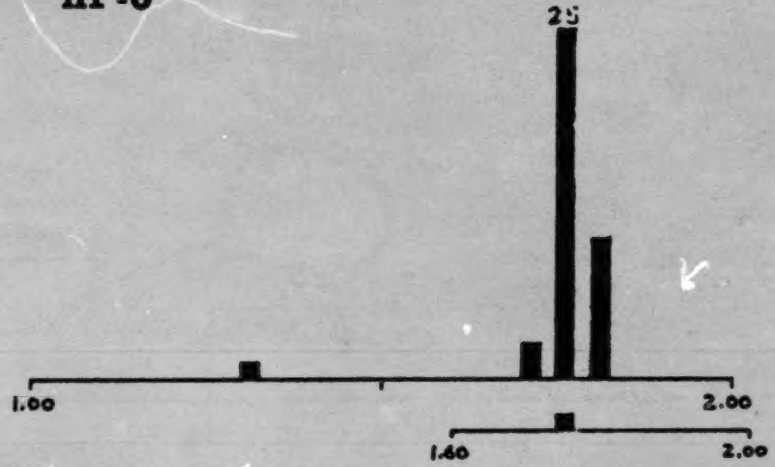
Time allowed: 12sec. Time allowed: 10sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	25	24	25	38	34	36	45	36	41	74	51	63
Discrimination Index	.36	.44	.39	.44	.45	.45	.53	.42	.50	.56	.37	.48

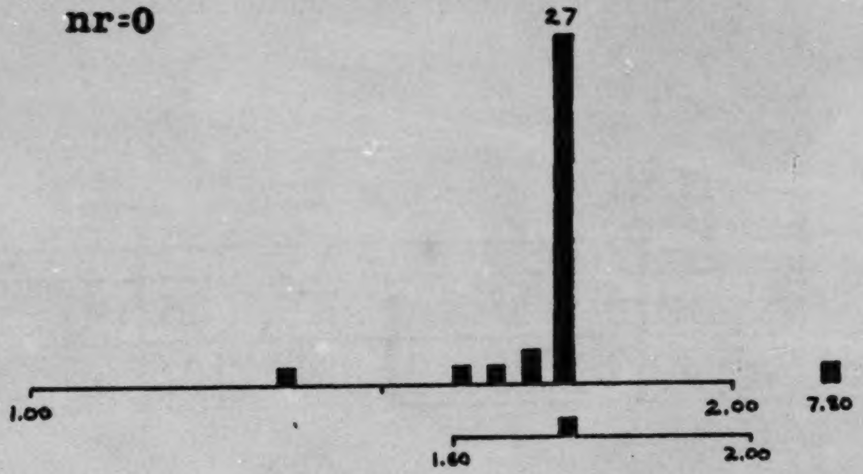
APPL 6



Grades 7,8
n=45 nr=0



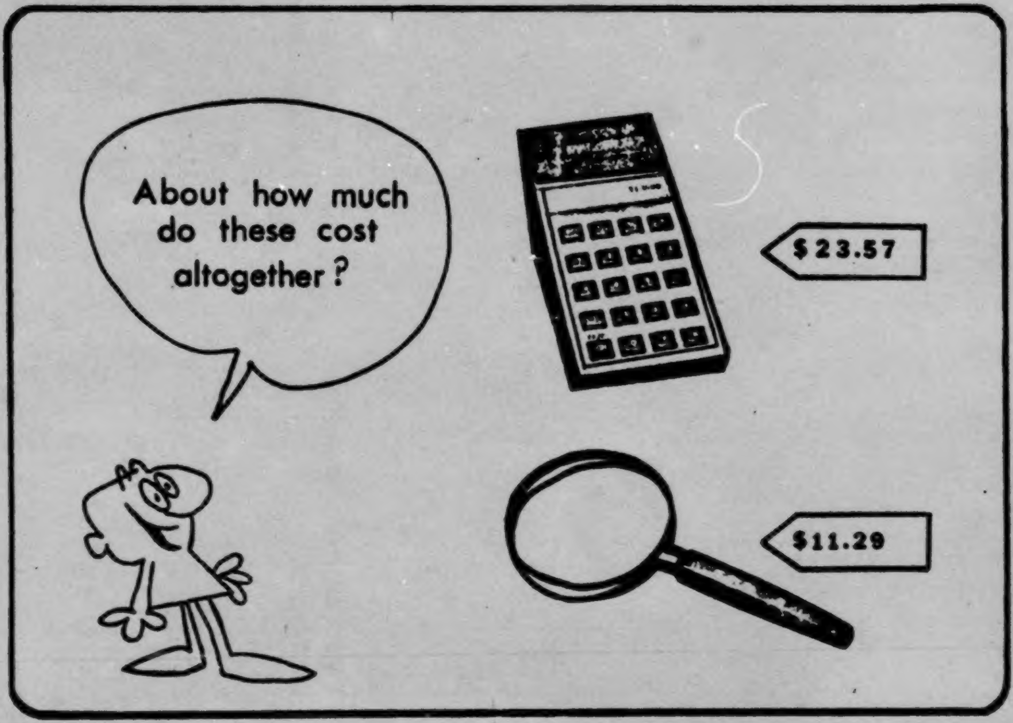
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 7


About how much do these cost altogether?
$23.57
$11.29

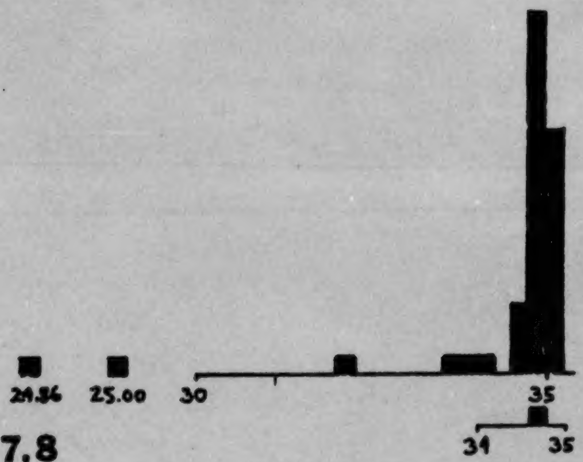
Acceptable Interval 34~35

Screening Data:

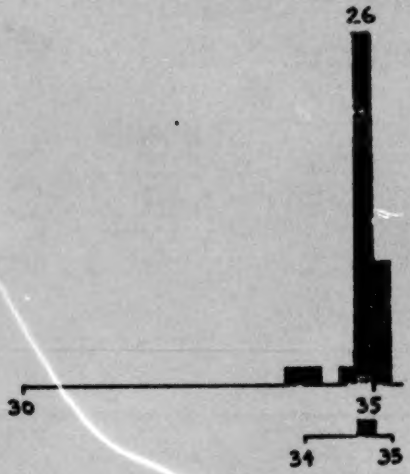
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	70	65	68	80	76	77	77	75	76	86	80	83
Discrimination Index	.51	.46	.49	.42	.35	.38	.33	.27	.31	.55	.09	.34

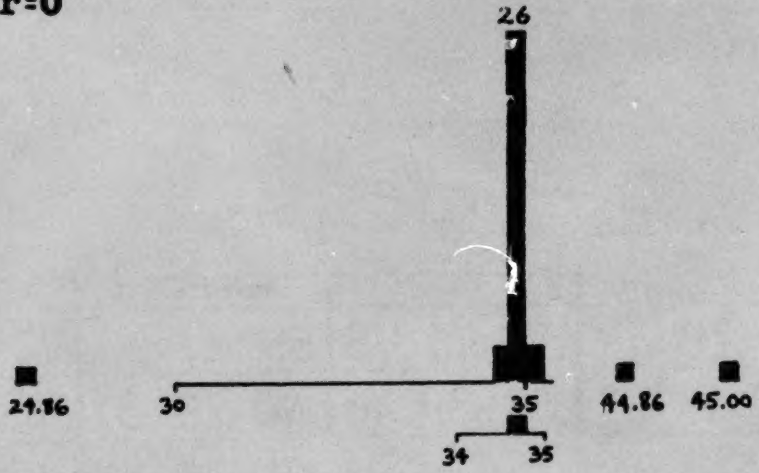
APPL 7



Grades 7,8
n=45 nr=0



Grades 9,10
n=36 nr=0

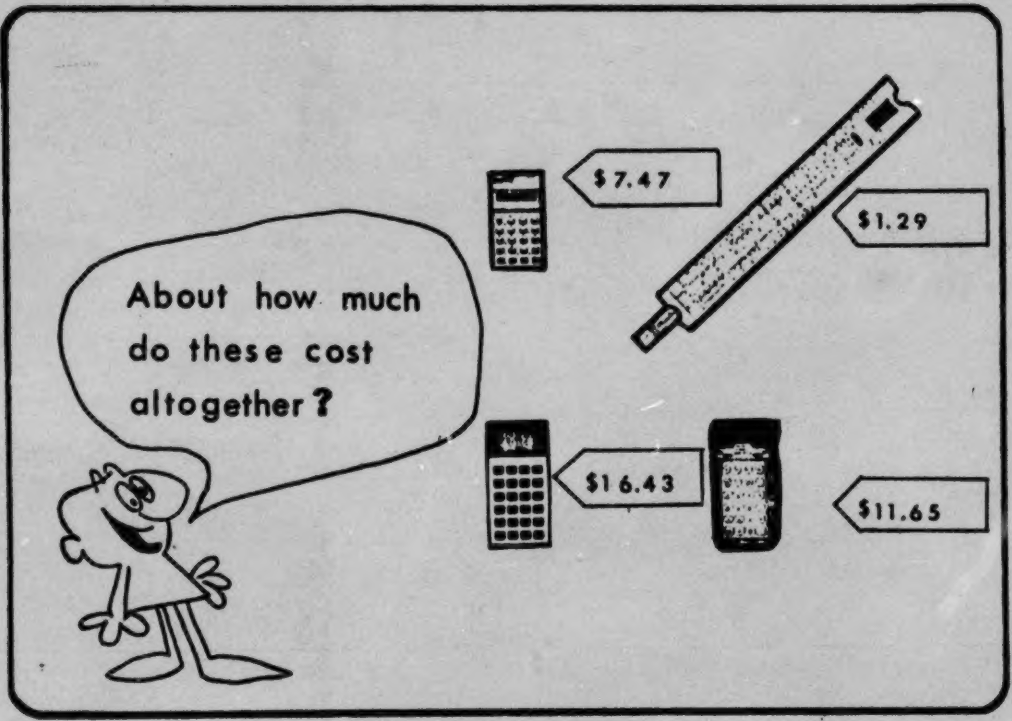


Grades 11,12
n=33 nr=0

Note: ■ represents one response

Item Analysis Summary for ACE Test

Exercise 8


About how much do these cost altogether?
$7.47
$1.29
$16.43
$11.65

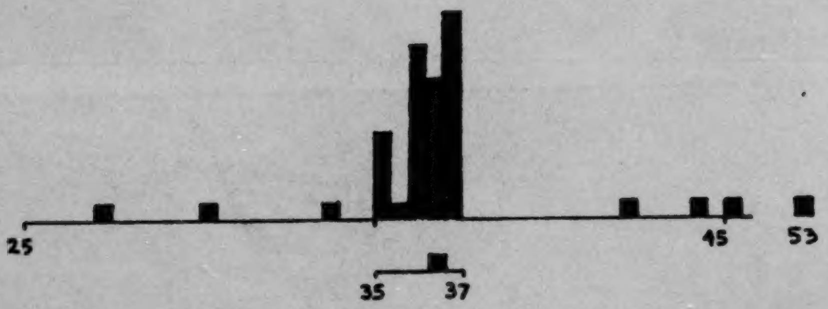
Acceptable Interval 35-37

Screening Data:

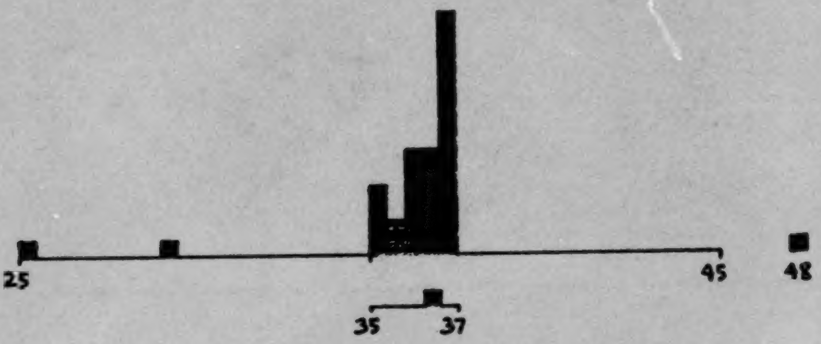
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	39	28	34	55	43	48	59	42	52	72	57	65
Discrimination Index	.50	.41	.47	.52	.39	.46	.40	.29	.37	.49	.17	.35

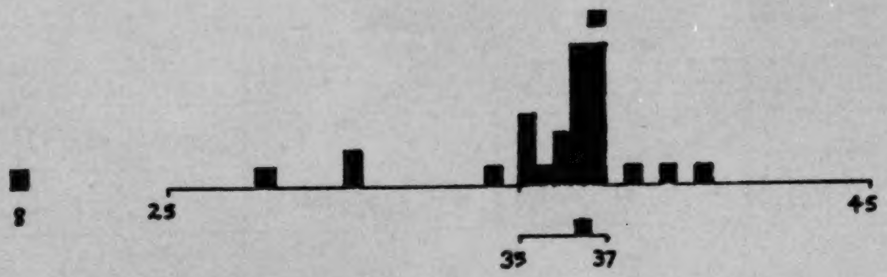
APPL 8



Grades 7,8
n=45 nr=2



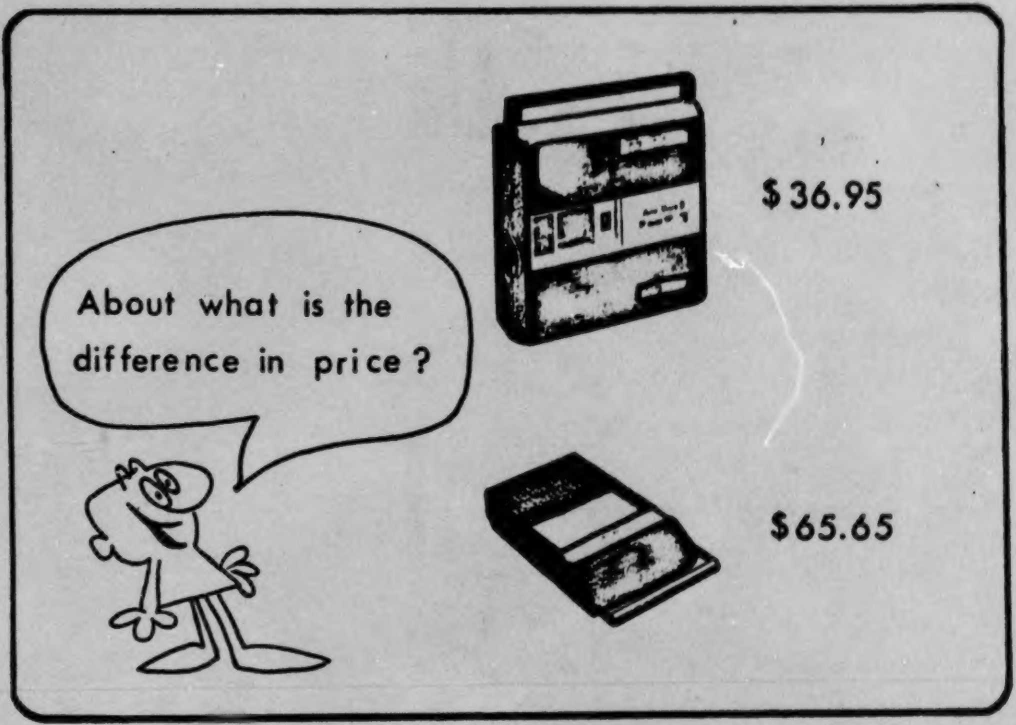
Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 9


About what is the difference in price?
$36.95
$65.65

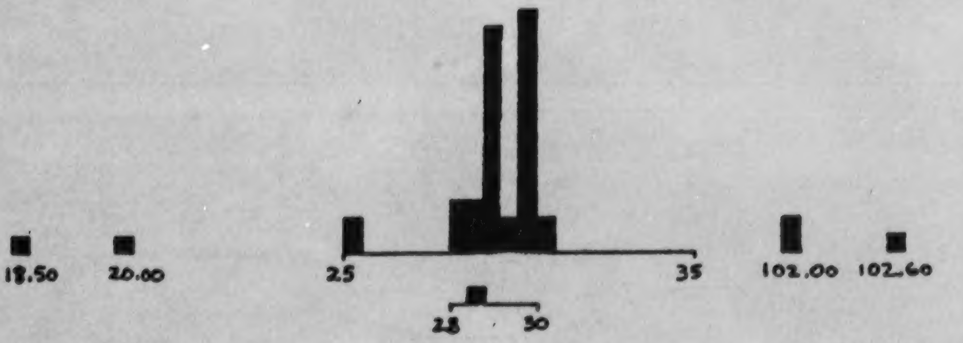
Acceptable Interval 28-30

Screening Data:

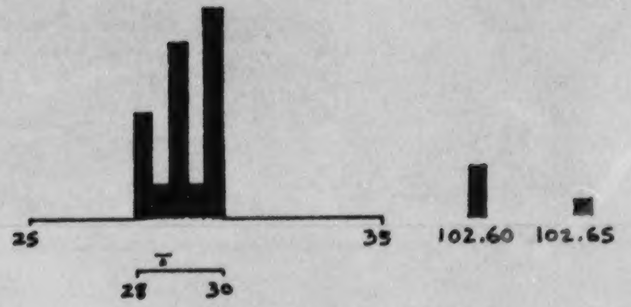
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	45	34	40	60	48	53	55	43	50	63	51	58
Discrimination Index	.52	.53	.53	.48	.41	.45	.48	.39	.45	.55	.37	.48

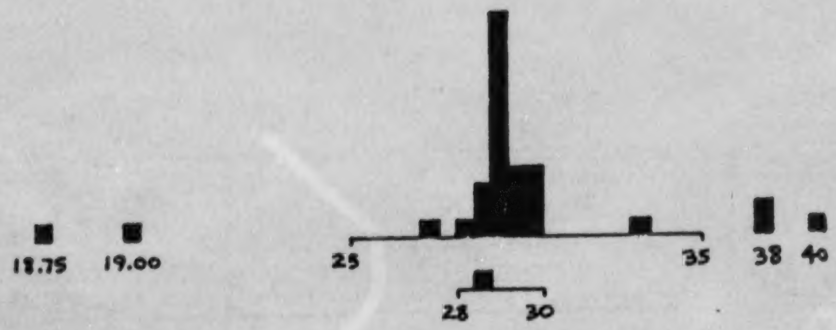
APPL 9



Grades 7,8
n=45 nr=1



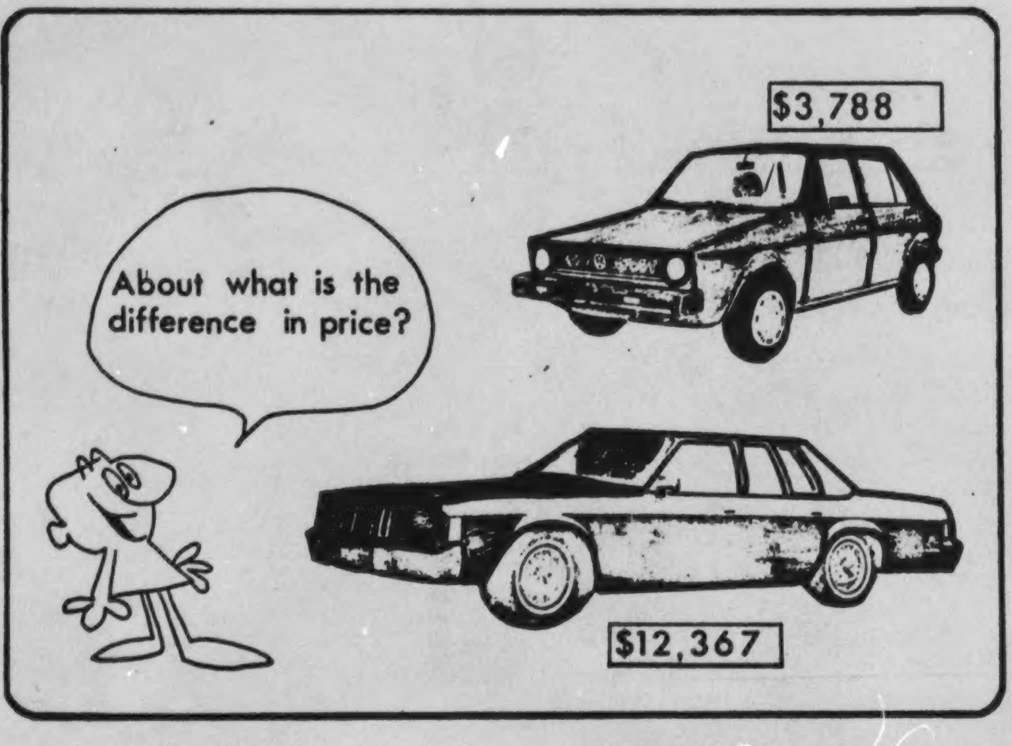
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise 10


$3,788
About what is the difference in price?
$12,367

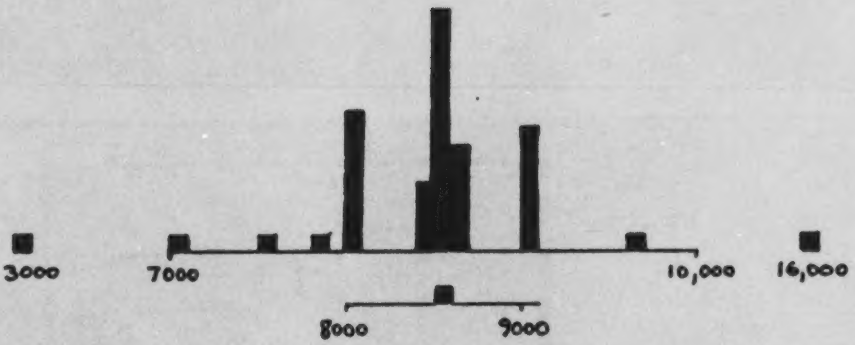
Acceptable Interval 8,000-9,000

Screening Data:

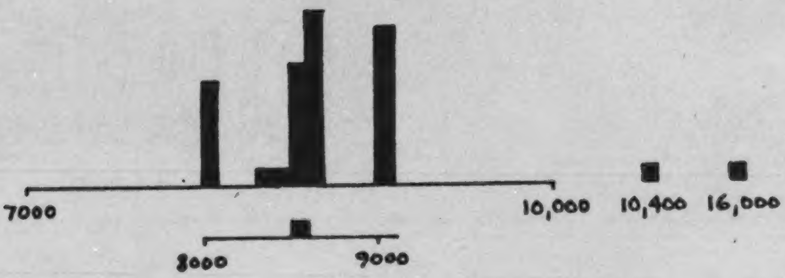
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (150)	F (209)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	54	41	48	71	62	66	64	56	61	70	73	72
Discrimination Index	.56	.55	.57	.42	.43	.43	.50	.49	.50	.62	.54	.57

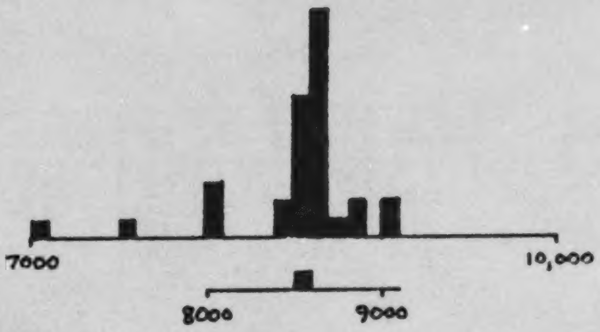
APPL 10



Grades 7,8
n=45 nr=0



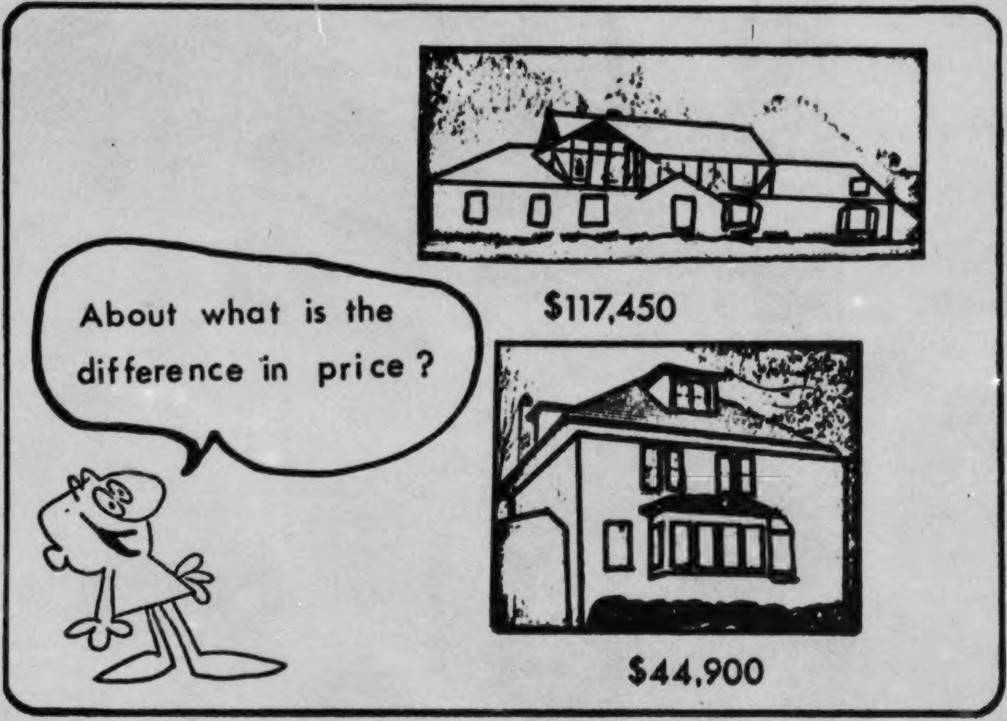
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 11


About what is the difference in price?
$117,450
$44,900

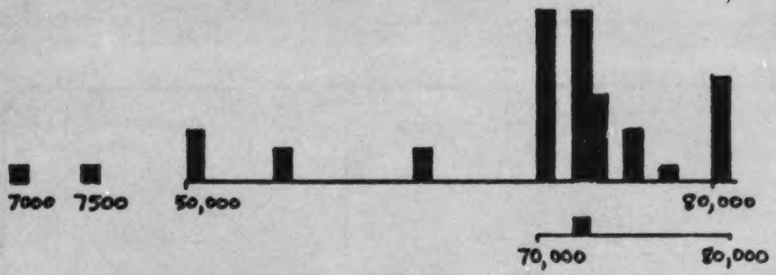
Acceptable Interval 70,000–80,000

Screening Data:

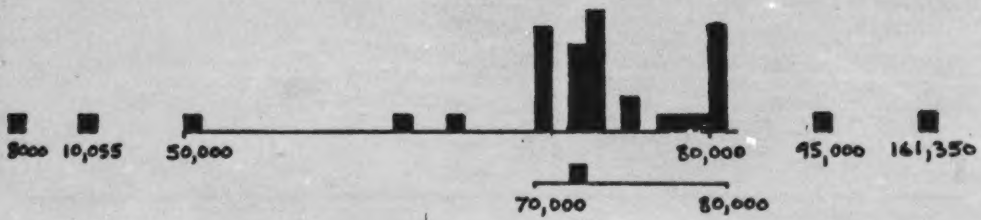
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	37	22	30	45	33	38	45	39	43	51	49	50
Discrimination Index	.63	.43	.56	.49	.43	.47	.48	.42	.46	.48	.49	.48

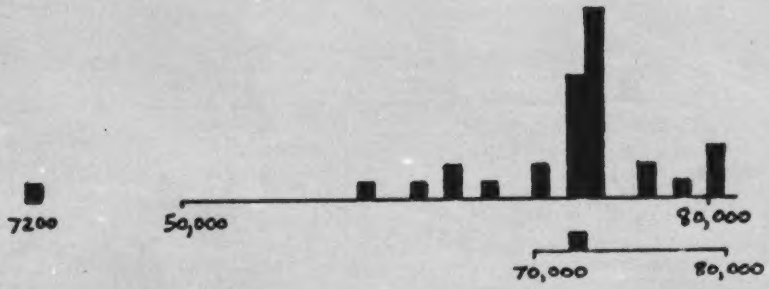
APPL 11



Grades 7,8
n=45 nr=1



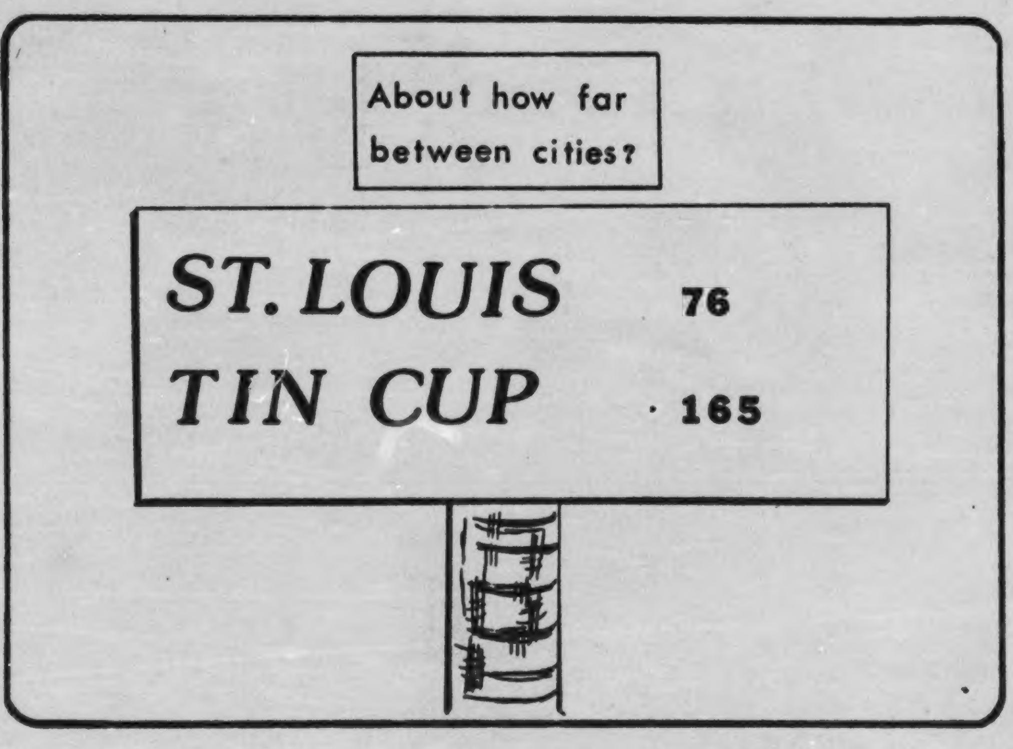
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 12


About how far between cities?
ST. LOUIS 76
TIN CUP 165

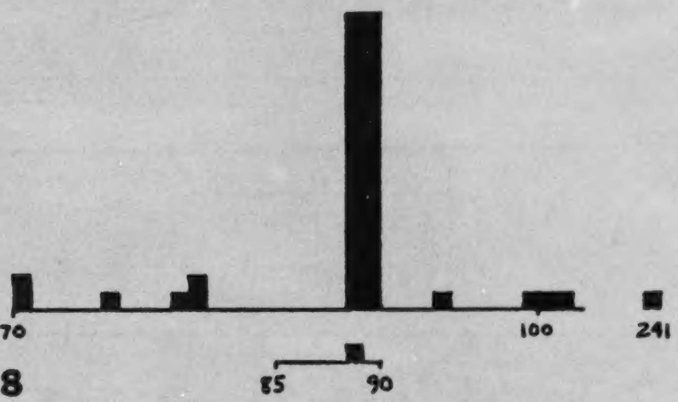
Acceptable Interval 85-90

Screening Data:

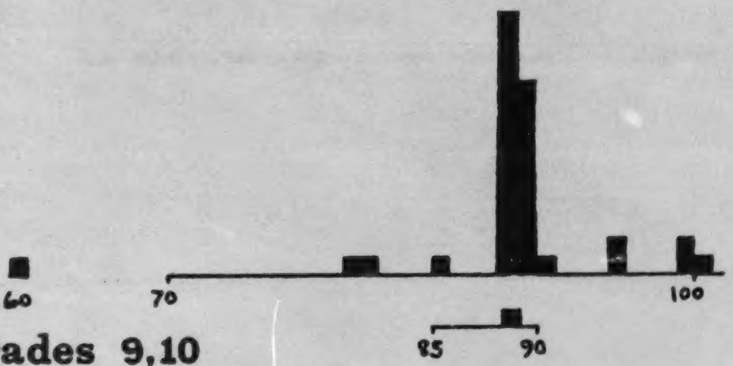
Time allowed: 12 sec.   Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	45	32	39	45	48	47	51	53	52	77	61	70
Discrimination Index	.56	.45	.52	.32	.30	.30	.36	.39	.37	.51	.43	.48

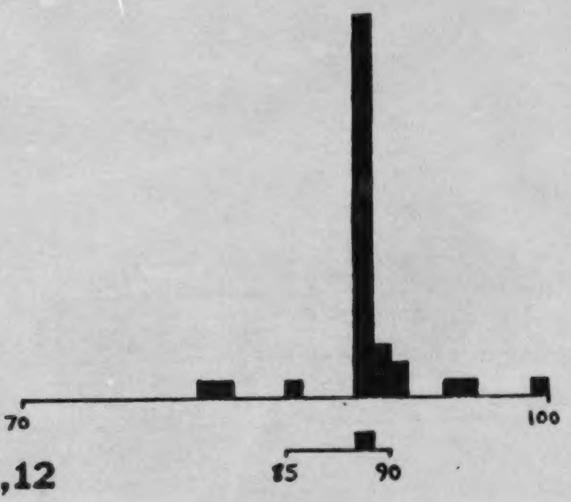
APPL 12



Grades 7,8
n=45 nr=0



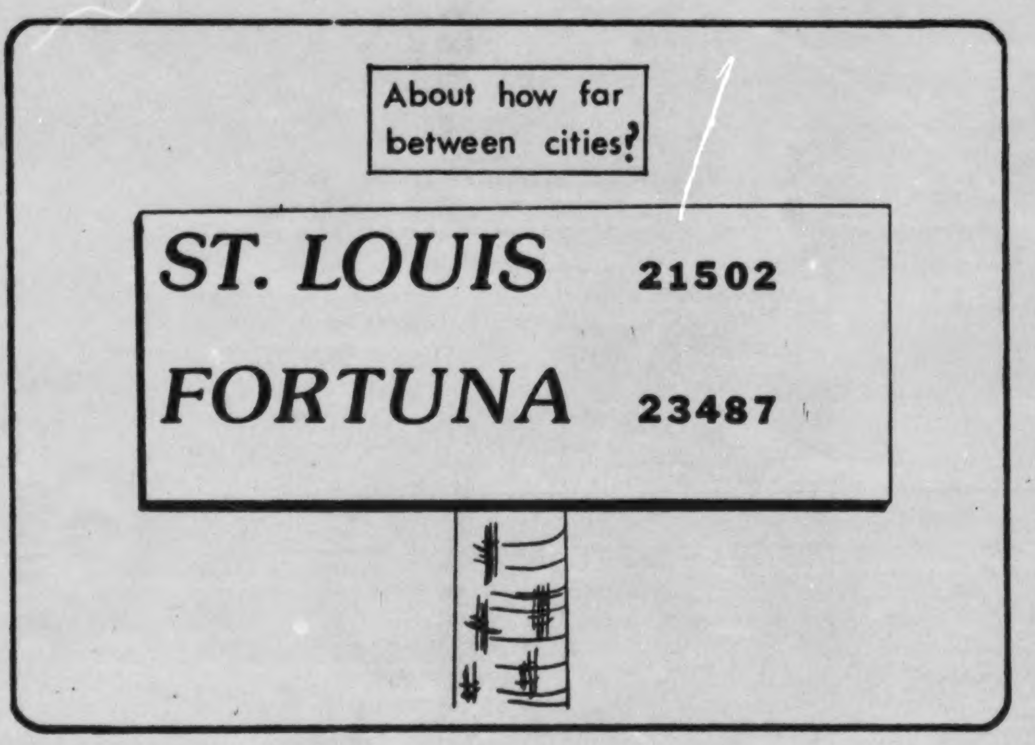
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■represents one response

Exercise **13**


About how far between cities?
ST. LOUIS 21502
FORTUNA 23487

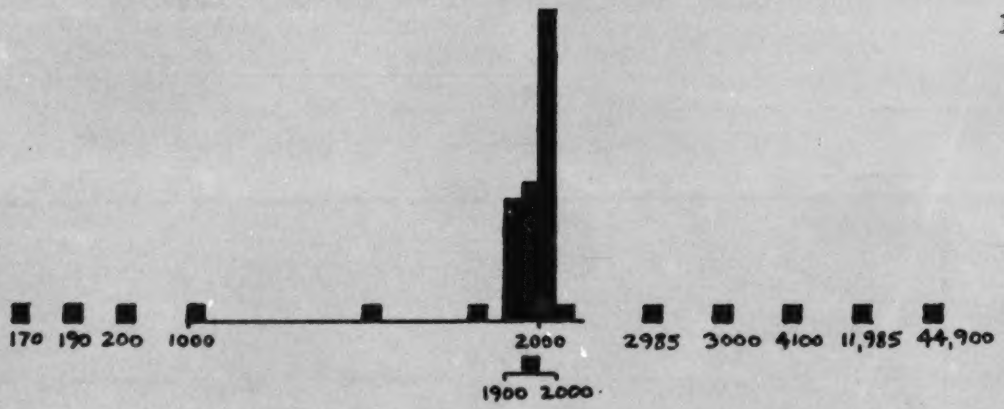
Acceptable Interval 1,900~2,000

Screening Data:

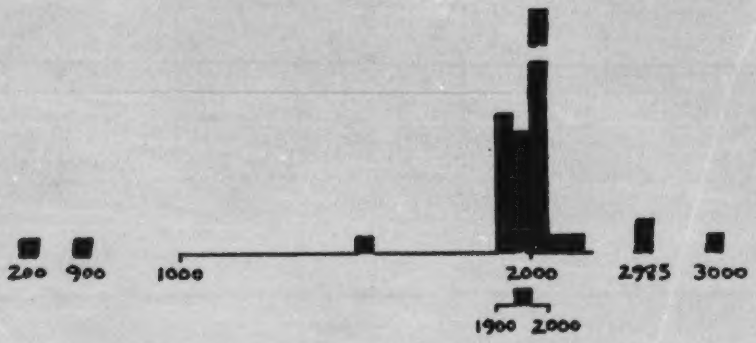
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	41	34	38	53	50	51	54	56	55	67	78	72
Discrimination Index	.45	.53	.49	.32	.37	.35	.47	.36	.42	.48	.42	.43

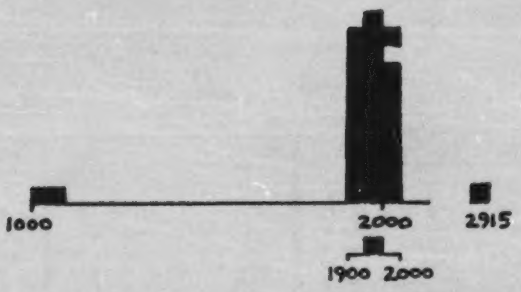
APPL 13



Grades 7,8
n=45 nr=0



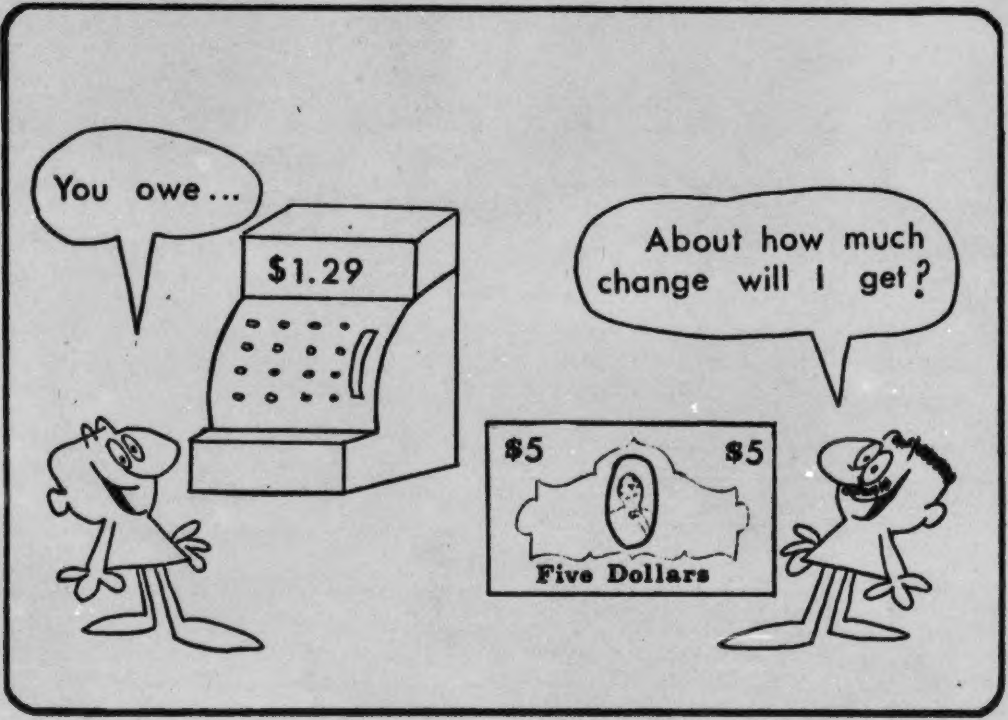
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 14


You owe...
$1.29
About how much change will I get?
$5
$5
Five Dollars

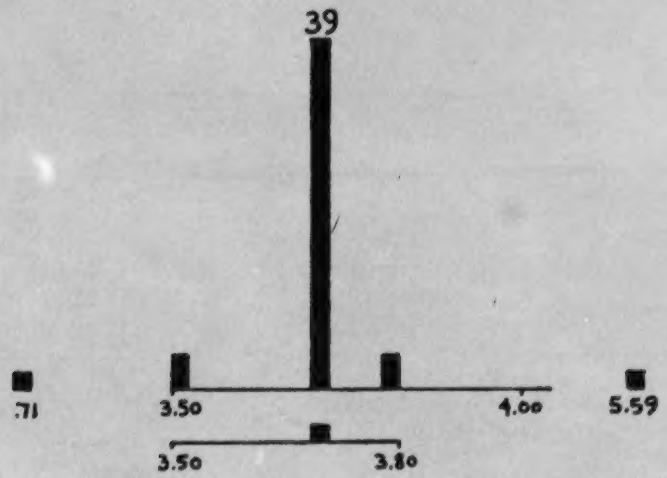
Acceptable Interval 3.50-3.80

Screening Data:

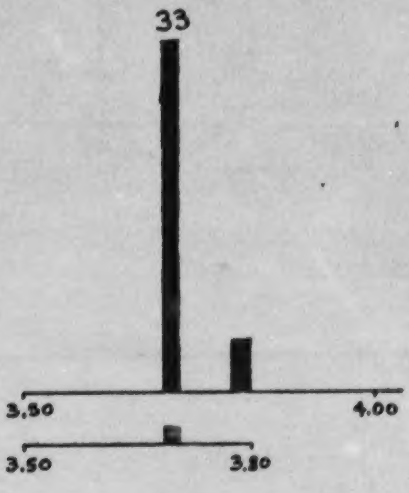
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9 -10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	66	45	56	85	71	77	85	75	81	84	84	84
Discrimination Index	.50	.48	.51	.43	.55	.51	.40	.32	.37	.59	.54	.56

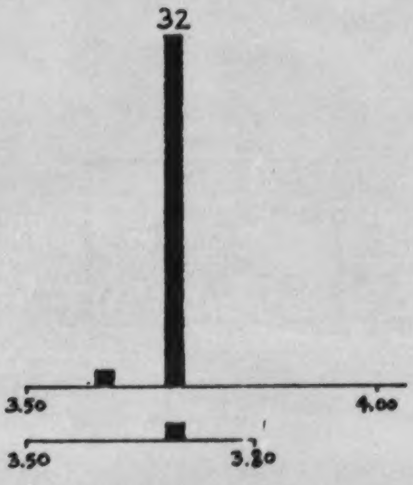
APPL 14



Grades 7,8
n=45 nr=0



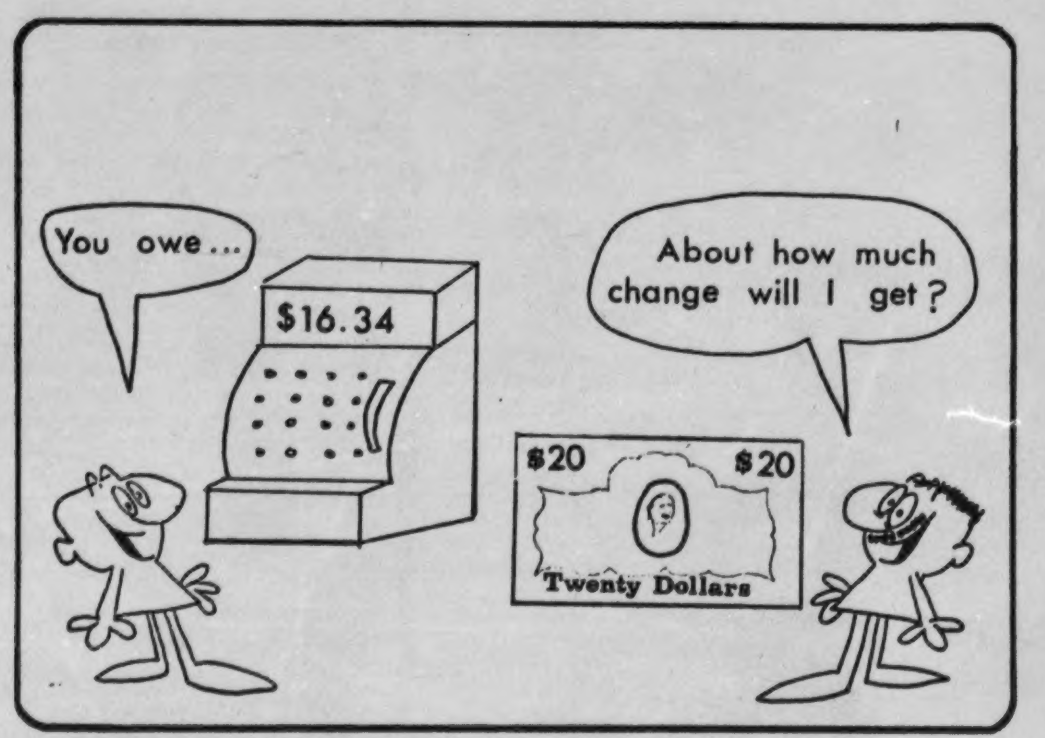
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise **15**


You owe...
$16.34
About how much change will I get?
$20 $20
Twenty Dollars

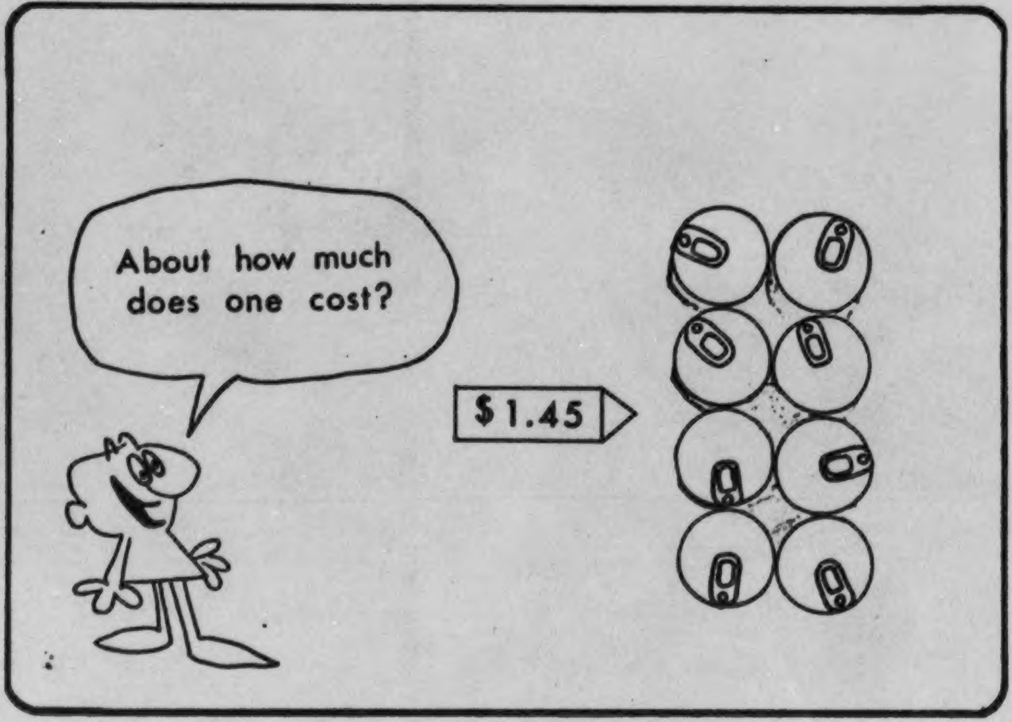
Acceptable Interval **3.50-4.00**

Screening Data:

Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209	T (431)	M (154)	F (205	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	66	52	59	82	82	82	87	82	85	86	90	88
Discrimination Index	.57	.55	.57	.56	.40	.46	.50	.48	.49	.59	.51	.54

Exercise 16


About how much does one cost?
$1.45

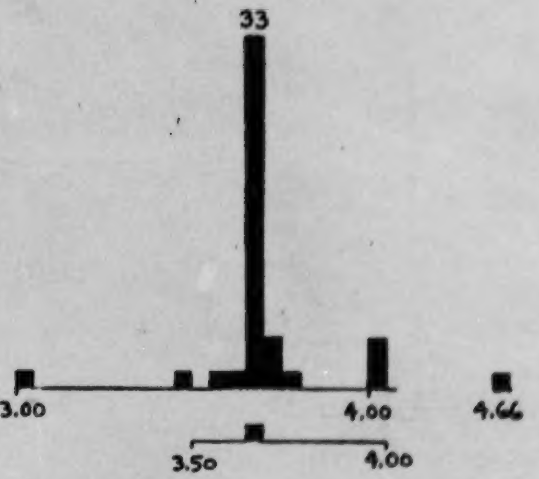
Acceptable Interval .15-.20

Screening Data:

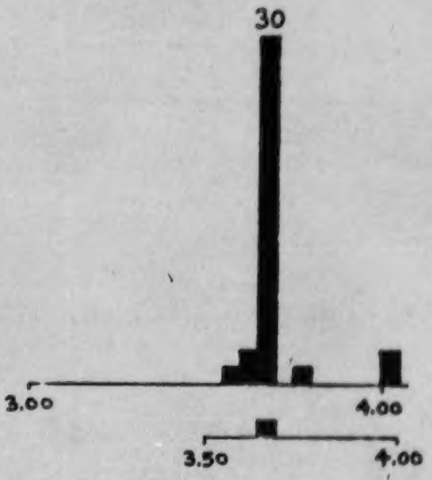
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	40	32	36	55	47	51	48	48	48	60	61	60
Discrimination Index	.57	.45	.52	.49	.44	.47	.46	.47	.46	.57	.67	.60

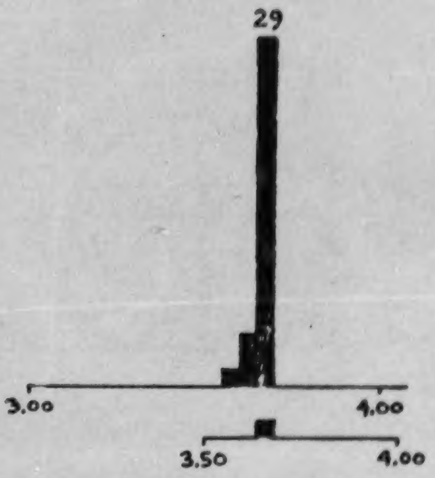
APPL 15



Grades 7,8
n=45 nr=0



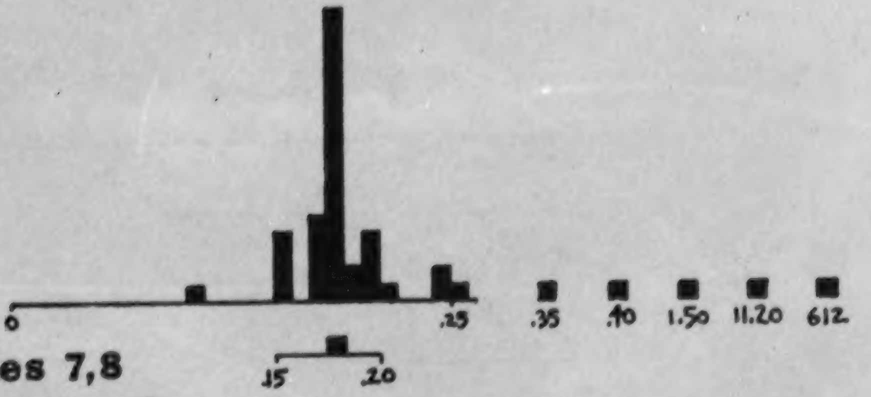
Grades 9,10
n=36 nr=0



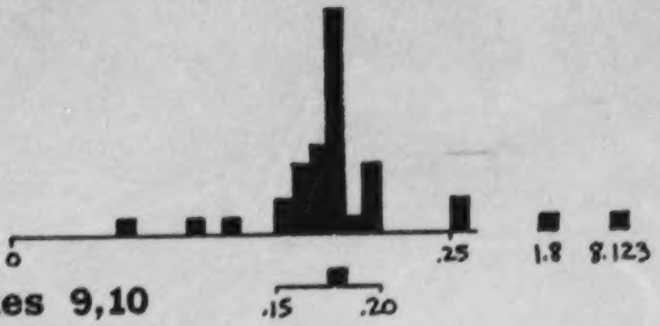
Grades 11,12
n=33 nr=0

Note: ■ represents one response

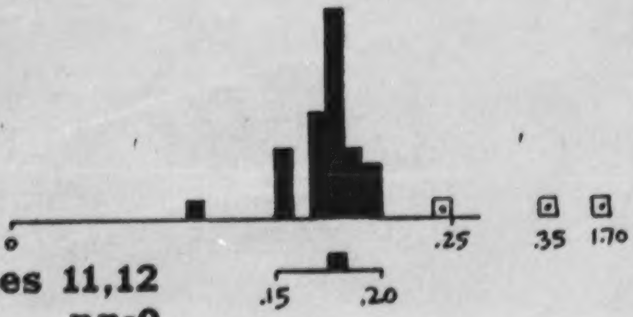
APPL 16



Grades 7,8
n=45 nr=2



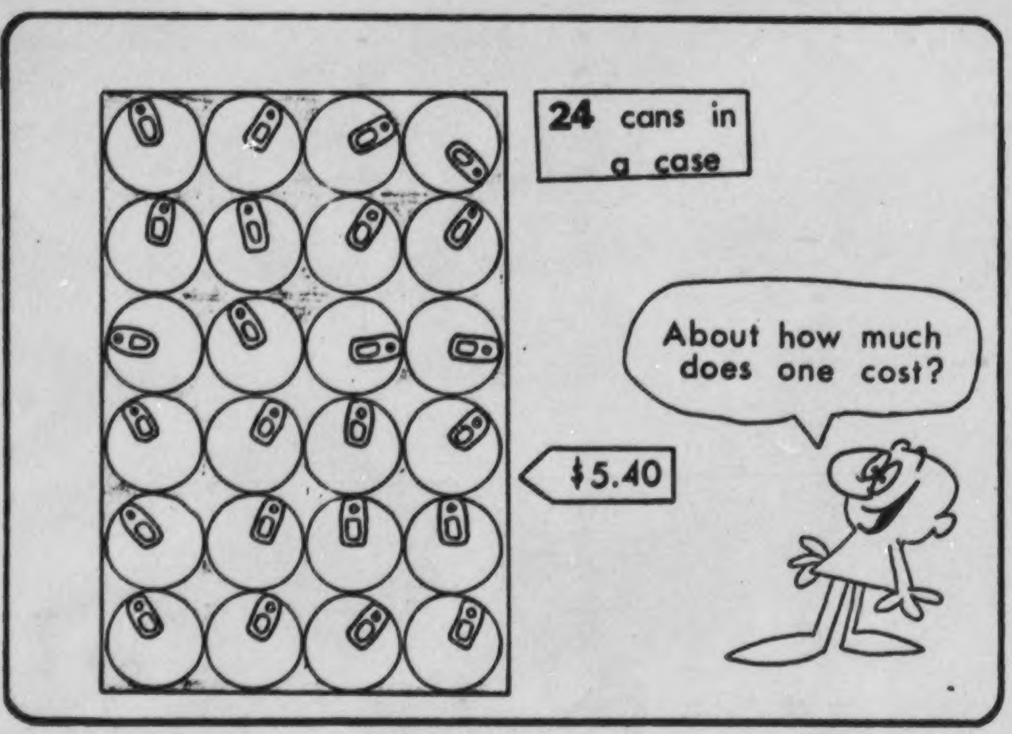
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 17


24 cans in a case
About how much does one cost?
$5.40

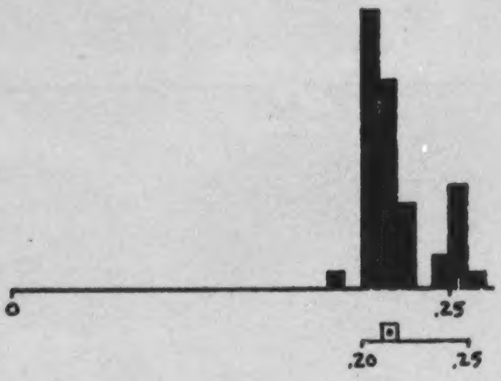
Acceptable Interval .20-.25

Screening Data:

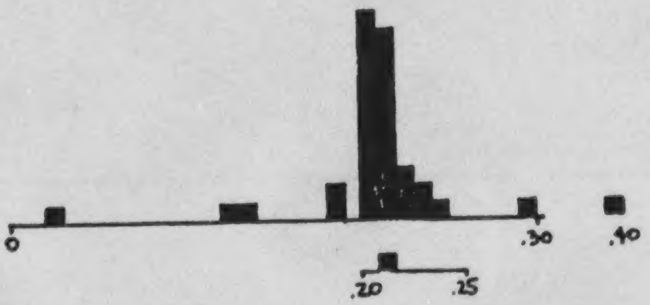
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	52	48	50	68	60	63	62	68	65	81	76	78
Discrimination Index	.57	.53	.55	.36	.39	.38	.51	.29	.42	.60	.63	.61

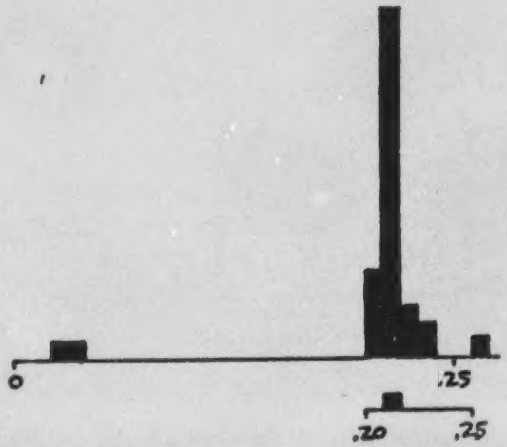
APPL 17



Grades 7,8
n=45 nr=0



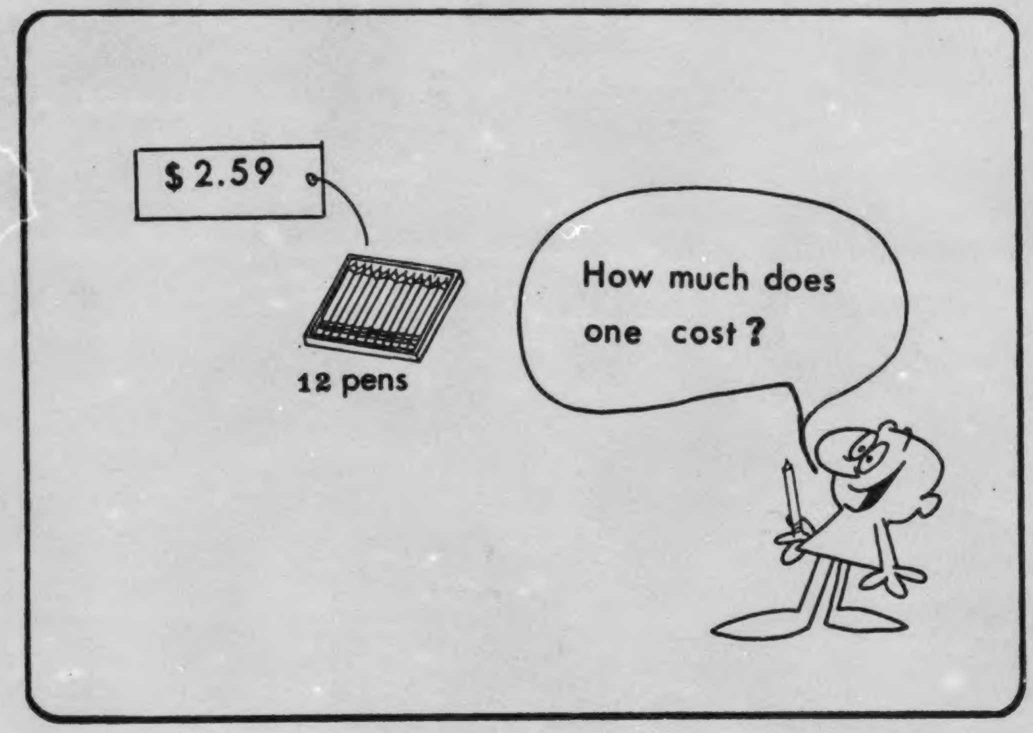
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 18


$ 2.59
12 pens
How much does one cost?

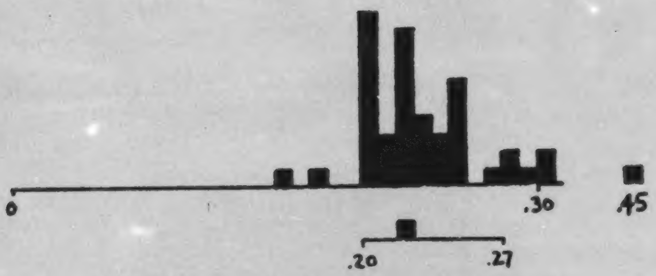
Acceptable Interval .20-.21

Screening Data:

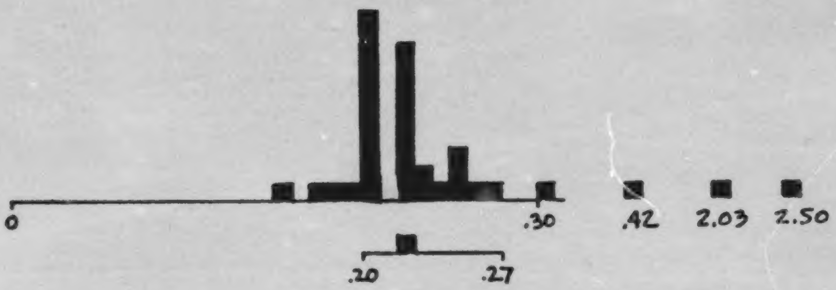
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T ([illegible])
Percent Correct	43	36	40	62	53	57	52	59	55	75	53	65
Discrimination Index	.58	.52	.56	.44	.39	.41	.51	.35	.44	.44	.40	.44

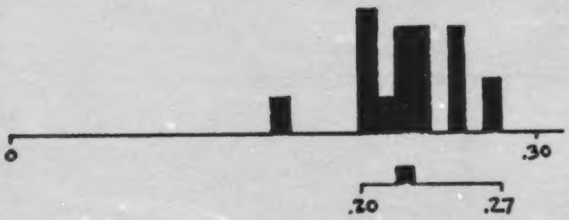
APPL 18



Grades 7,8
n=45 nr=1



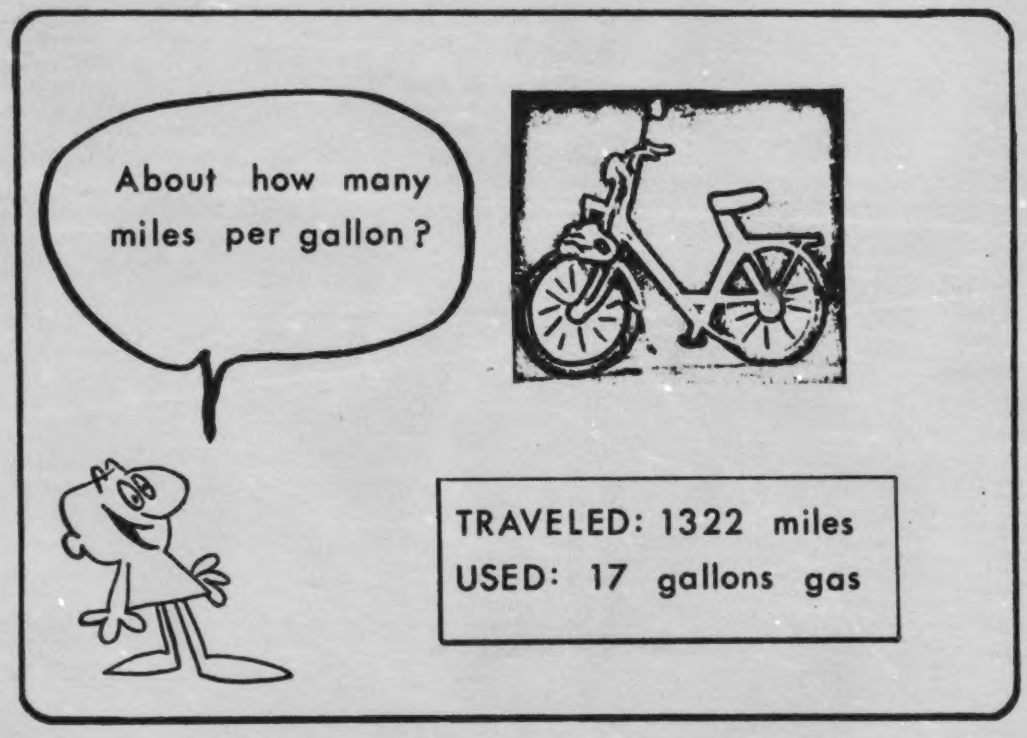
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise 19


About how many miles per gallon?
TRAVELED: 1322 miles
USED: 17 gallons gas

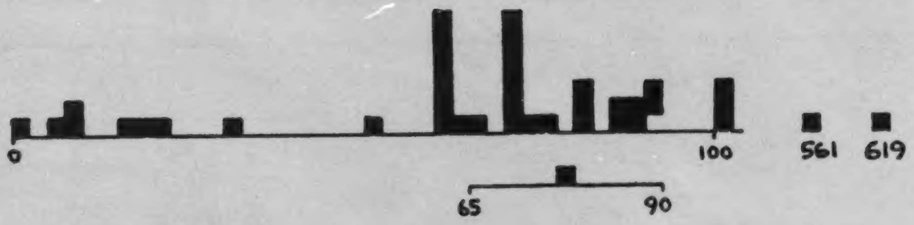
Acceptable Interval 65-90

Screening Data:

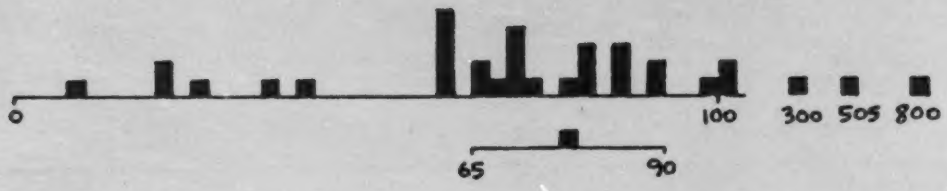
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	22	12	17	37	18	26	44	28	37	40	16	29
Discrimination Index	.31	.27	.31	.36	.31	.36	.45	.35	.41	.39	.47	.44

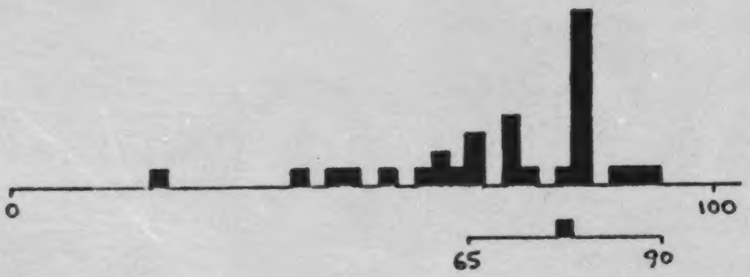
APPL 19



Grades 7,8
n=45 nr=4



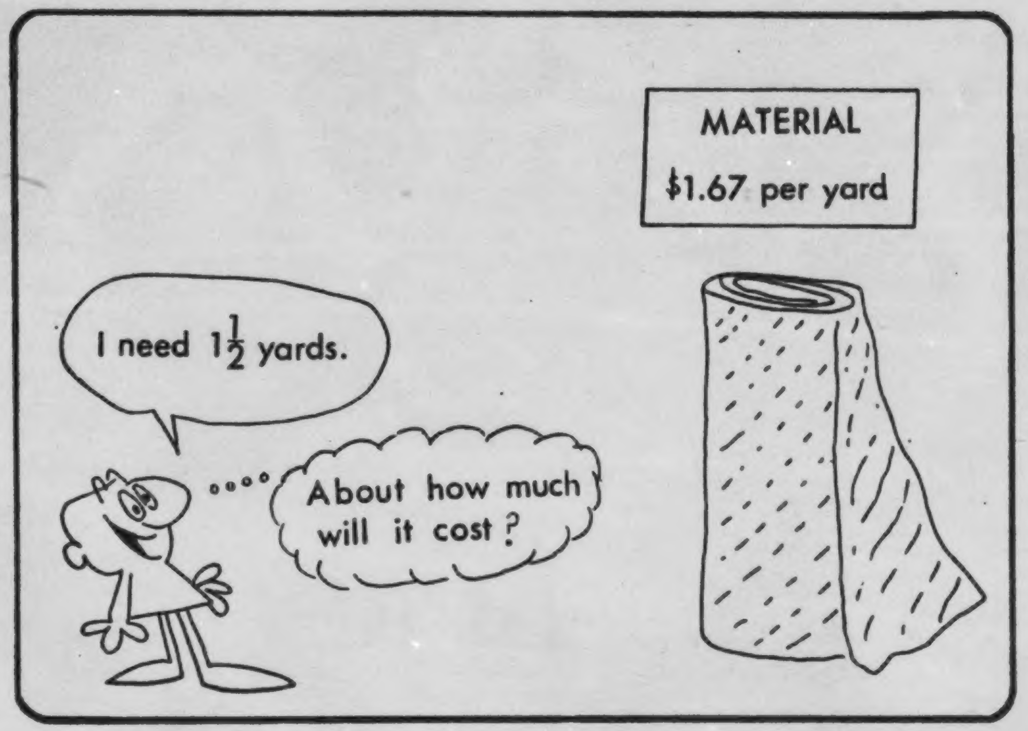
Grades 9,10
n=36 nr=2



Grades 11,12
n=33 nr=2

Note: ■ represents one response

Exercise 20


MATERIAL
$1.67 per yard
I need 1½ yards.
About how much will it cost?

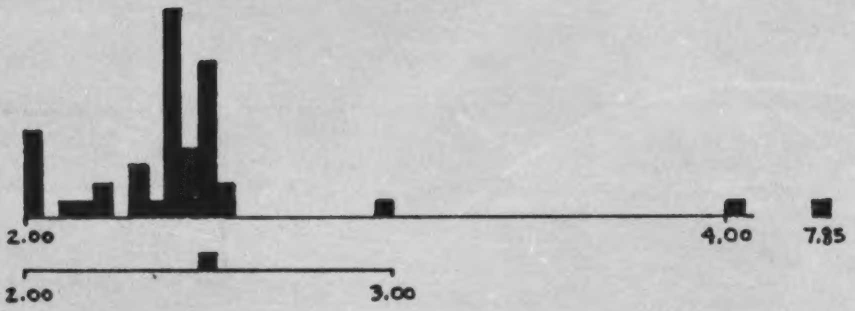
Acceptable Interval 2.00-3.00

Screening Data:

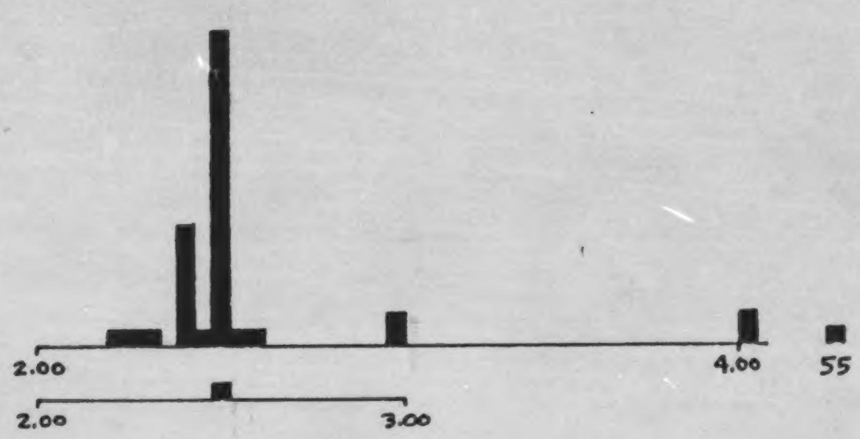
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	57	50	54	75	75	75	71	71	71	84	78	81
Discrimination Index	.46	.58	.52	.27	.37	.32	.41	.38	.39	.57	.35	.47

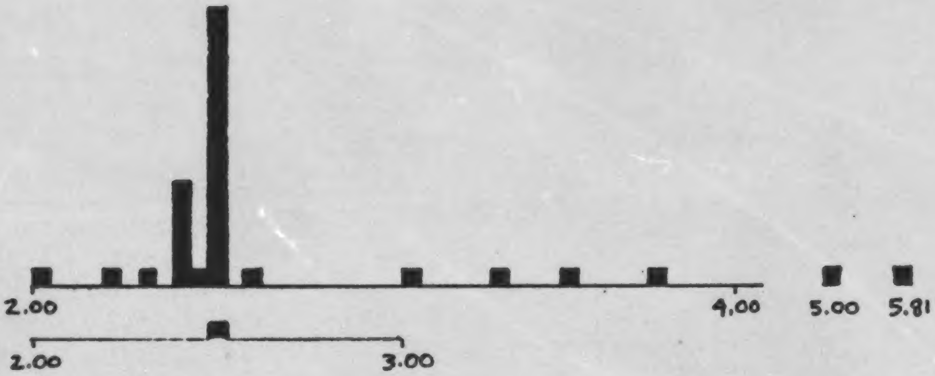
APPL 20



Grades 7,8
n=45 nr=2



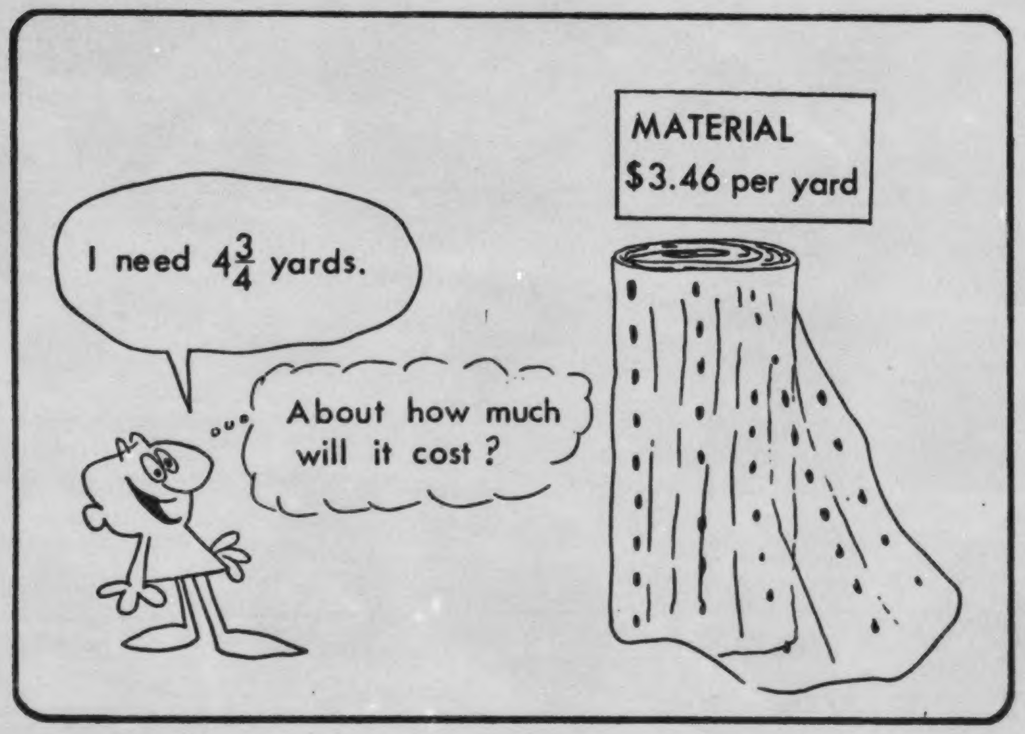
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise **21**


MATERIAL
$3.46 per yard
I need 4¾ yards.
About how much will it cost?

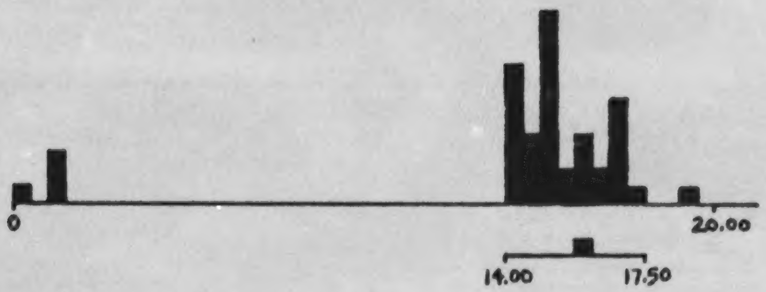
Acceptable Interval 14.00-17.50

Screening Data:

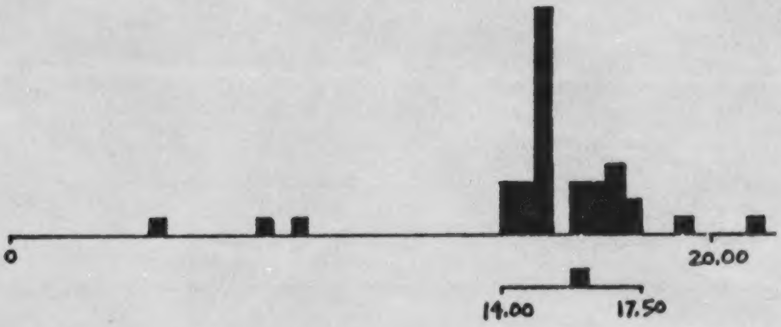
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	41	25	34	62	51	55	54	59	56	68	71	70
Discrimination Index	.55	.56	.57	.50	.41	.46	.49	.51	.49	.48	.29	.38

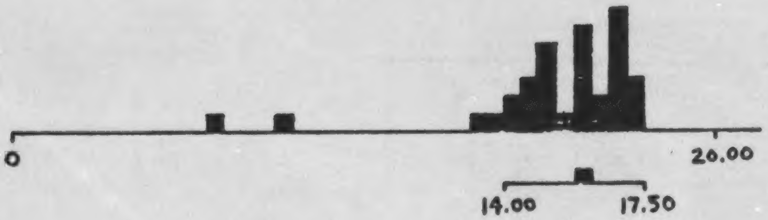
APPL 21



Grades 7,8
n=45 nr=2



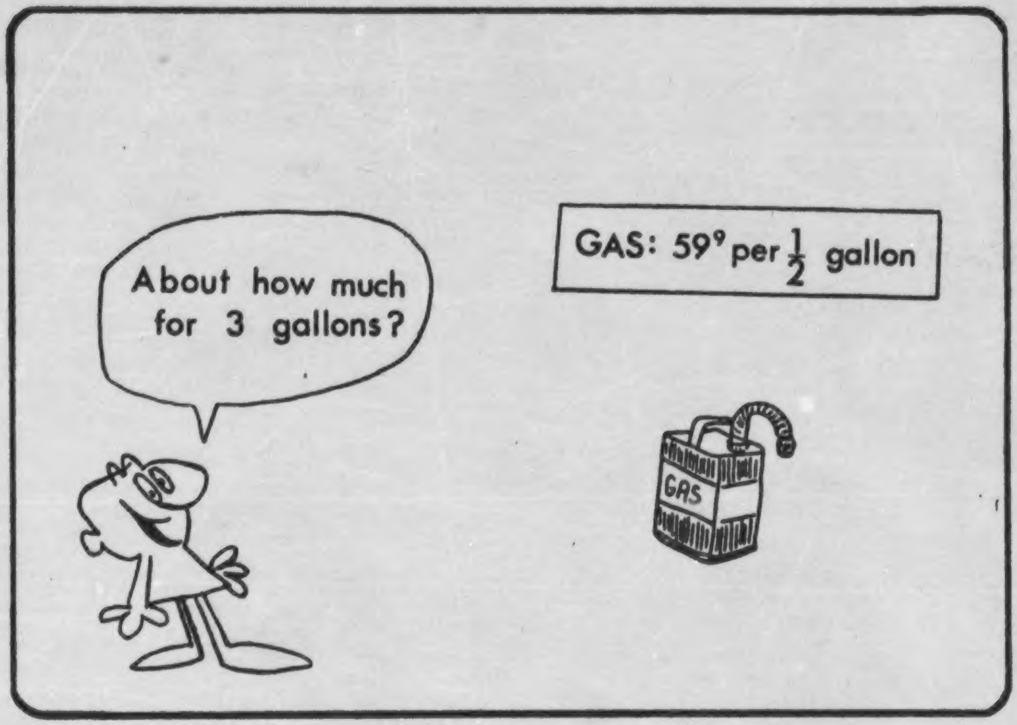
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 22


About how much for 3 gallons?
GAS: 59⁹ per ½ gallon
GAS

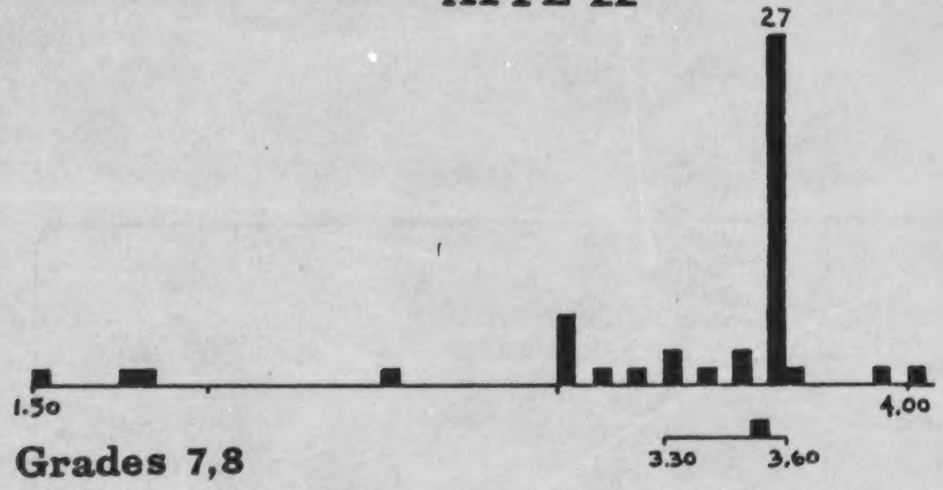
Acceptable Interval 3.30~3.60

Screening Data:

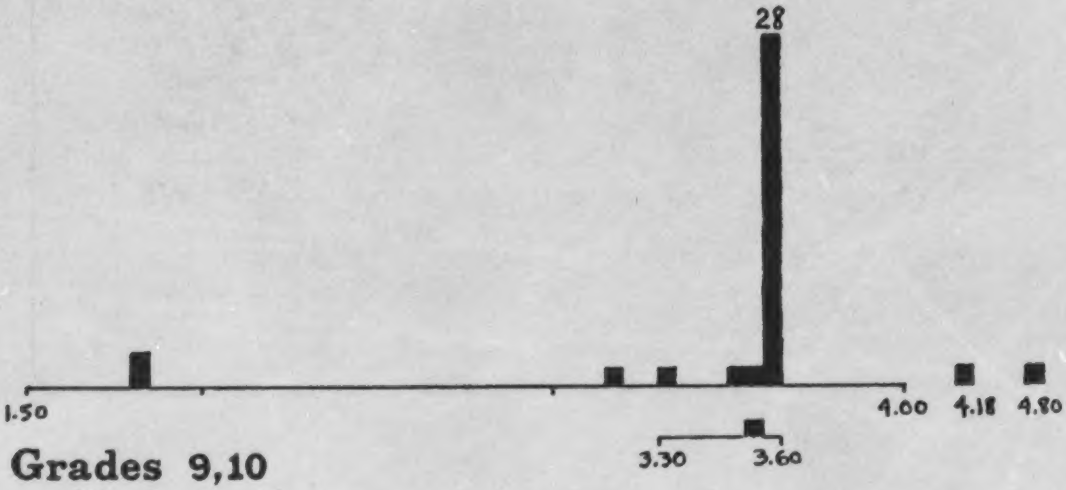
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	32	22	27	51	34	41	45	40	43	72	55	64
Discrimination Index	.58	.56	.58	.51	.43	.49	.49	.41	.46	.34	.65	.50

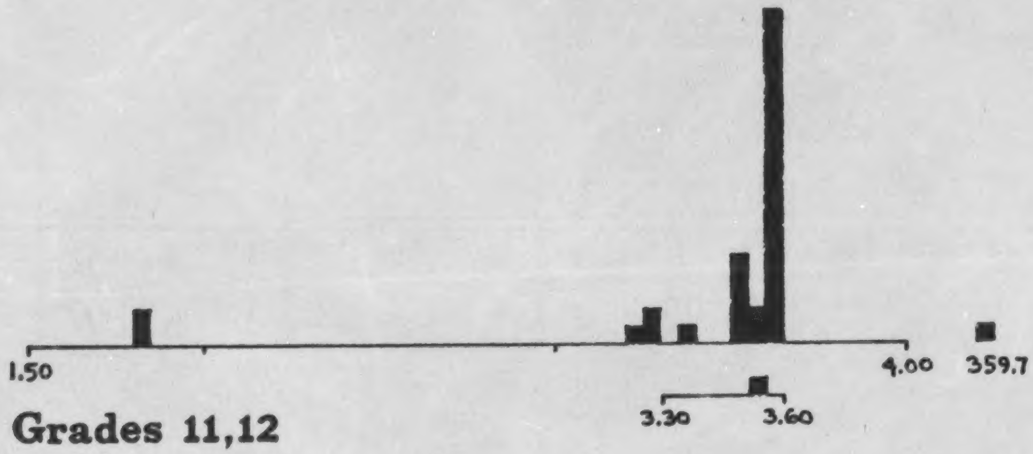
APPL 22



Grades 7,8
n=45 nr=0



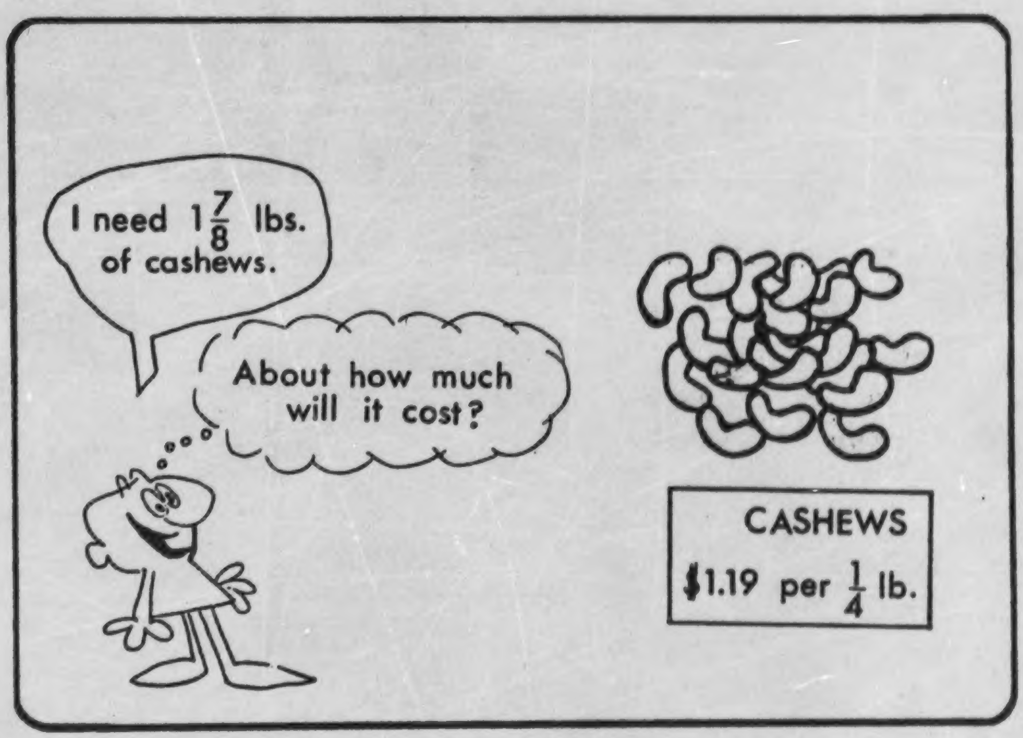
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 23


I need $1\frac{7}{8}$ lbs. of cashews.
About how much will it cost?
CASHEWS
$1.19 per $\frac{1}{4}$ lb.

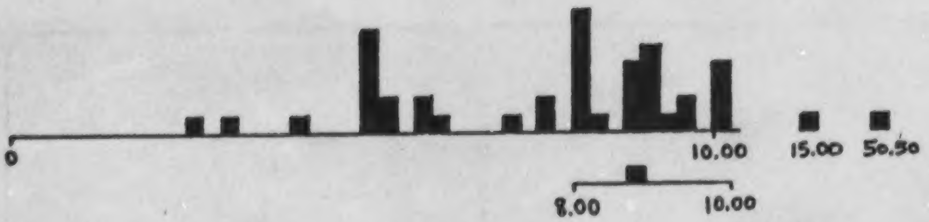
Acceptable Interval 8-10

Screening Data:

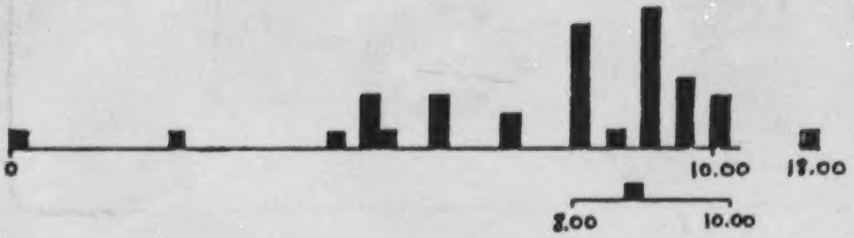
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	18	14	16	29	15	21	30	21	26	35	37	36
Discrimination Index	.39	.28	.35	.39	.34	.38	.50	.45	.49	.34	.53	.46

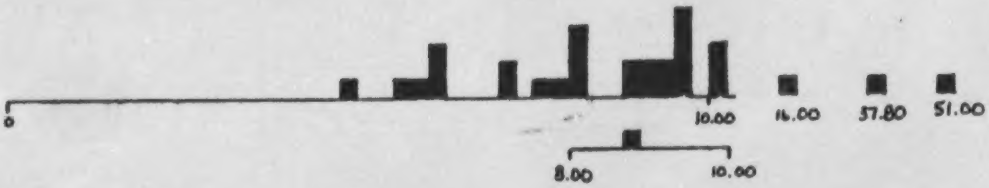
APPL 23



Grades 7,8
n=45 nr=2



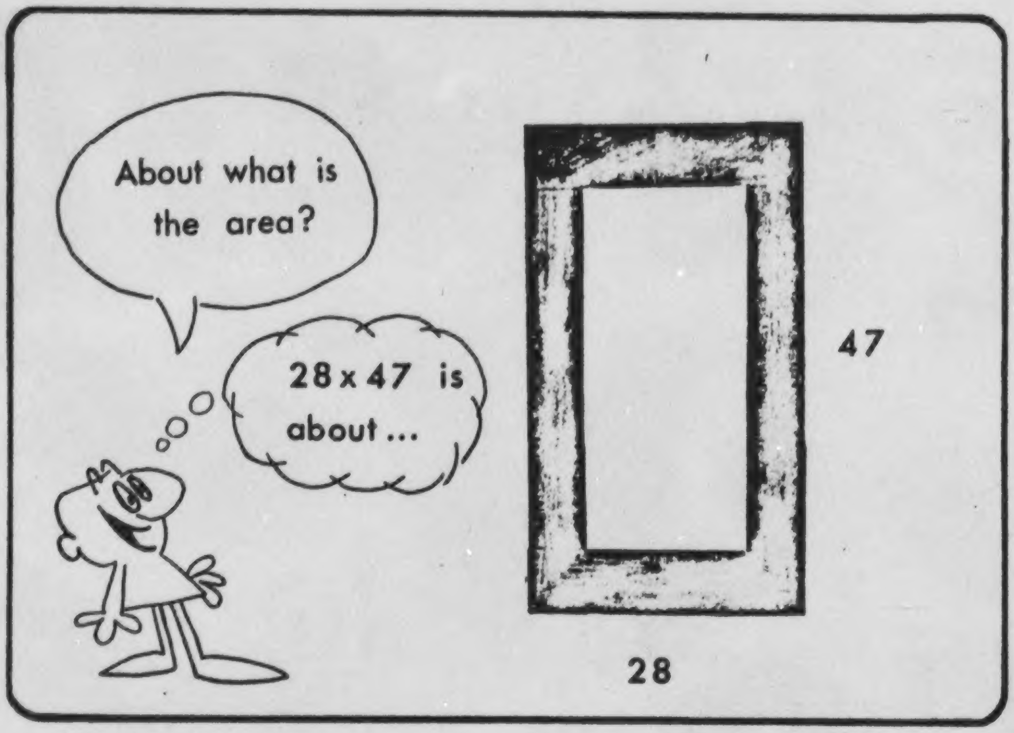
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=1

Note: ■ represents one response

Exercise 24


About what is the area?
28 x 47 is about ...
47
28

Acceptable Interval 1200-1500

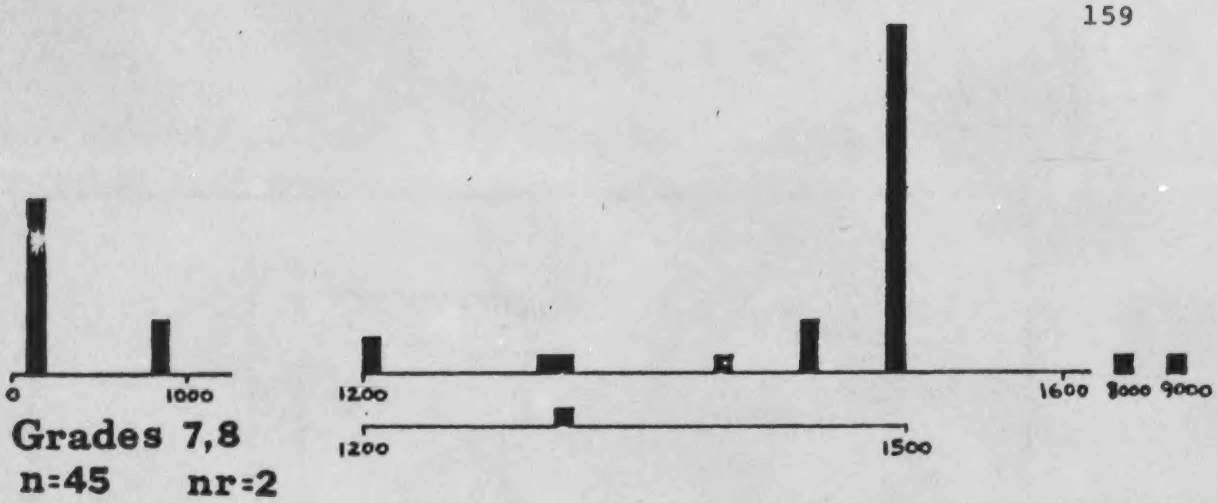
Screening Data:

Time allowed: 12 sec.

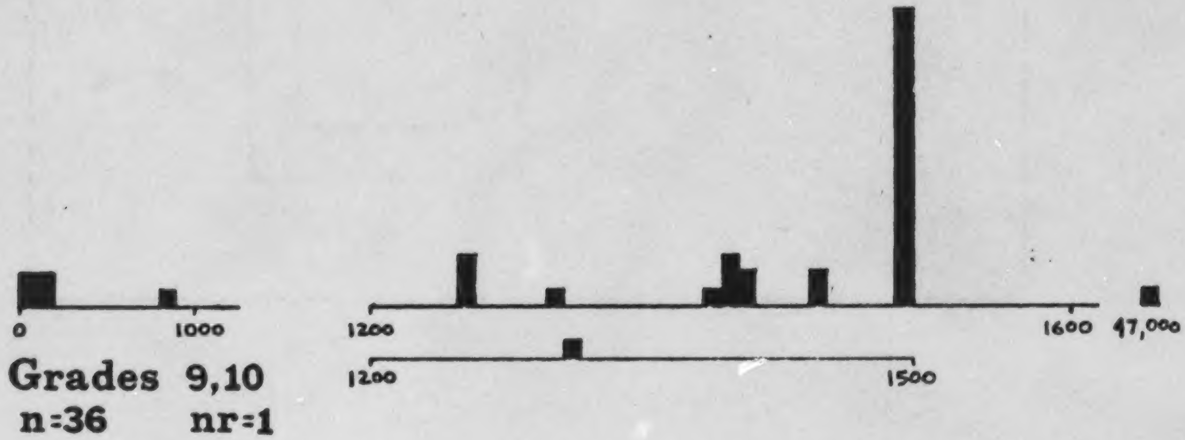
Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	21	16	19	40	28	33	36	32	35	61	51	57
Discrimination Index	.42	.41	.42	.61	.44	.53	.55	.30	.45	.50	.53	.52

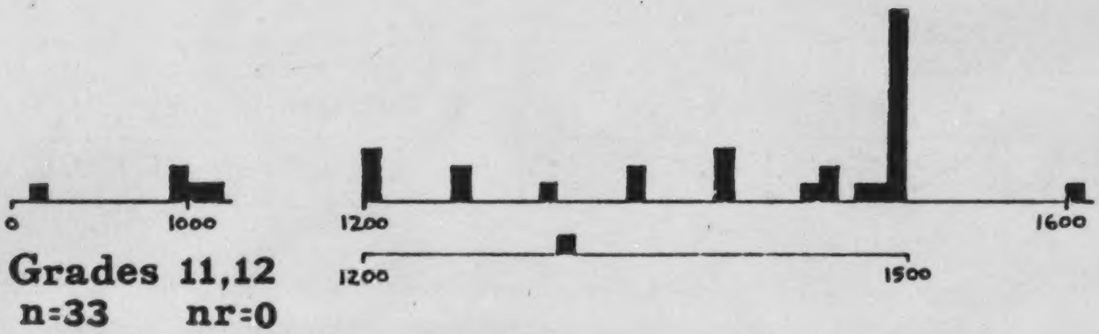
APPL 24



Grades 7,8
n=45 nr=2



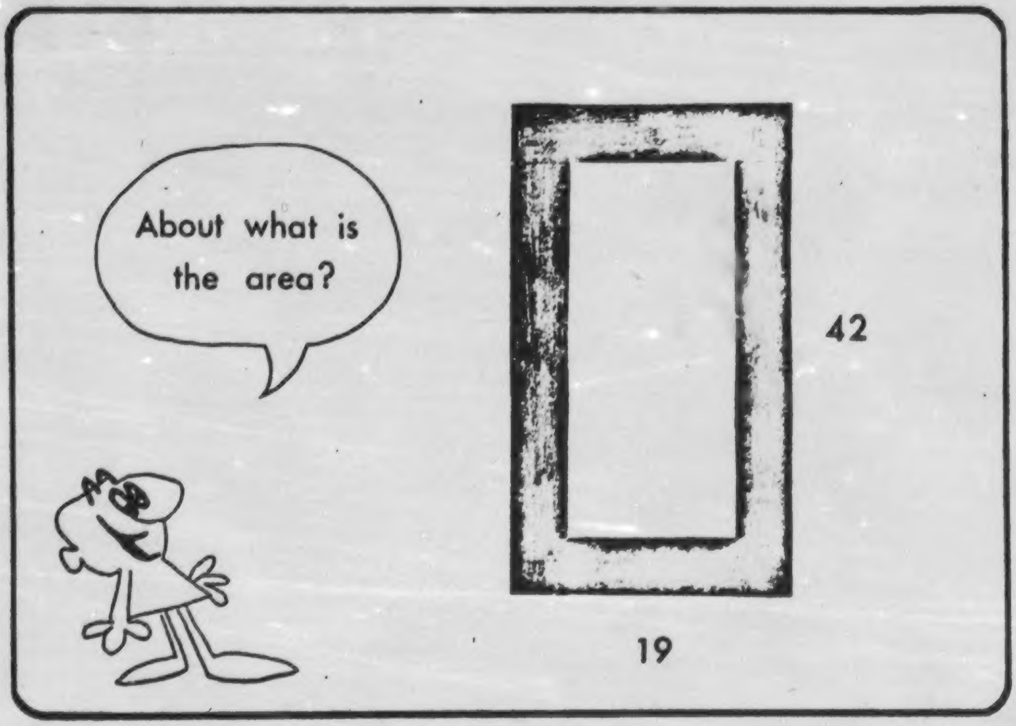
Grades 9,10
n=36 nr=1



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 25


About what is the area?
42
19

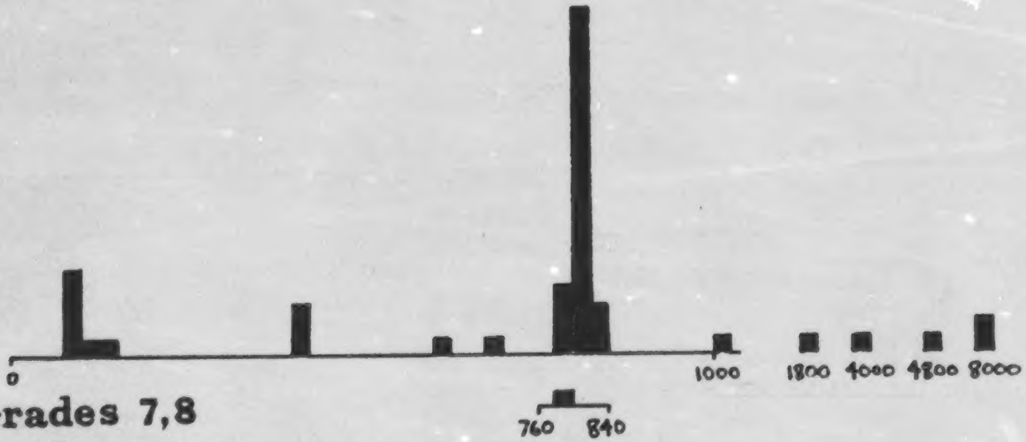
Acceptable Interval 760-840

Screening Data:

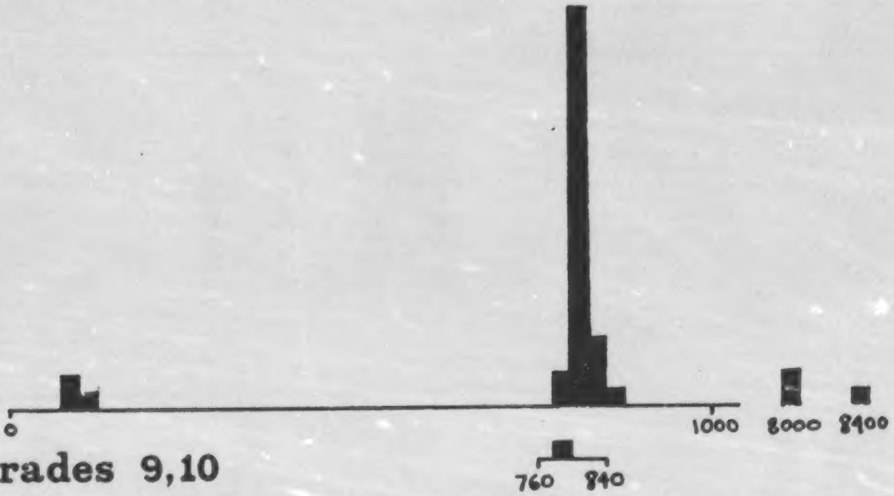
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7 - 8			Grade 9 - 10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	26	19	23	48	39	43	53	40	48	74	65	70
Discrimination Index	.45	.52	.48	.54	.46	.50	.55	.32	.47	.46	.61	.54

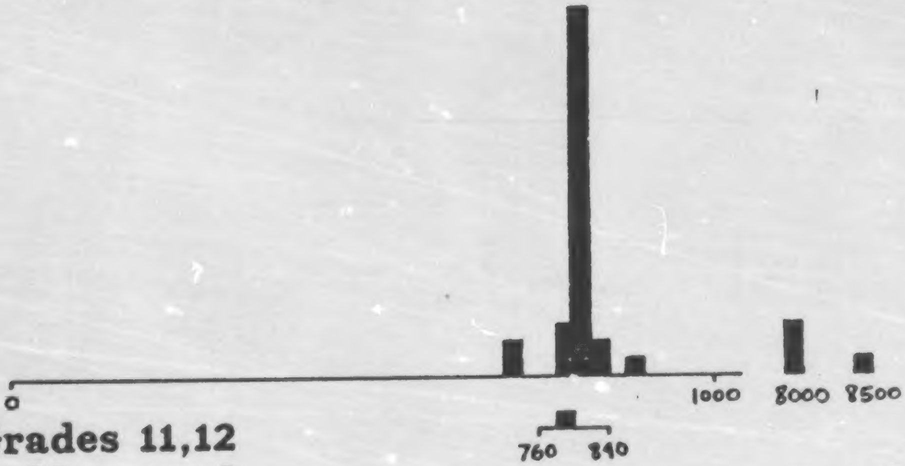
APPL 25



Grades 7,8
n=45 nr=0



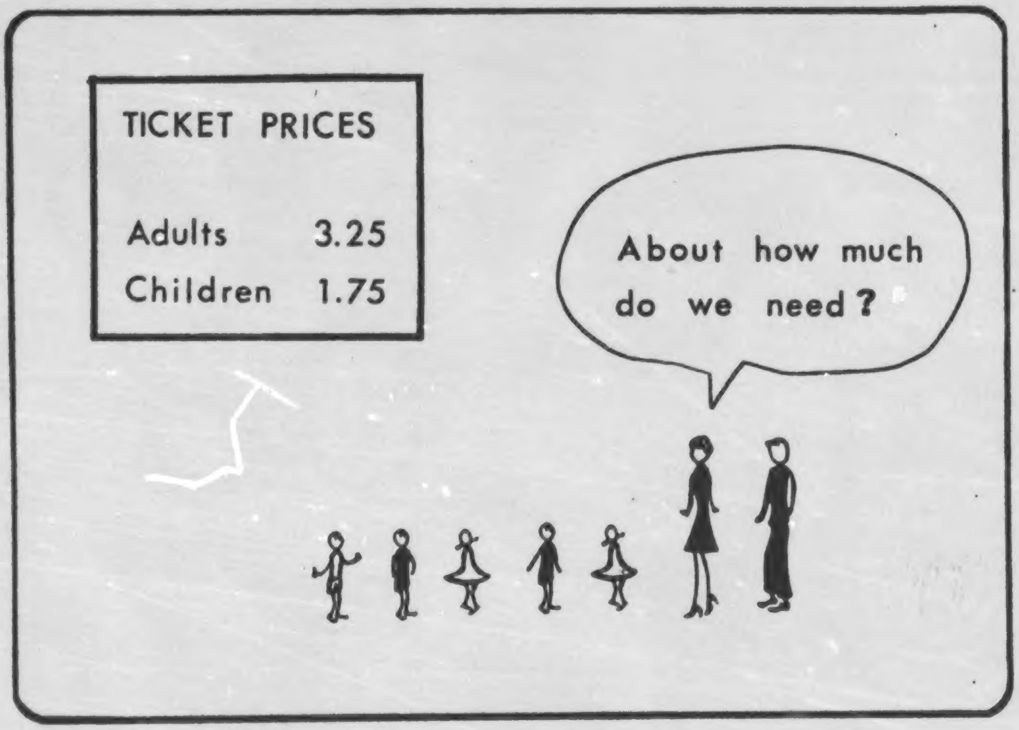
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 26


TICKET PRICES
Adults 3.25
Children 1.75
About how much do we need?

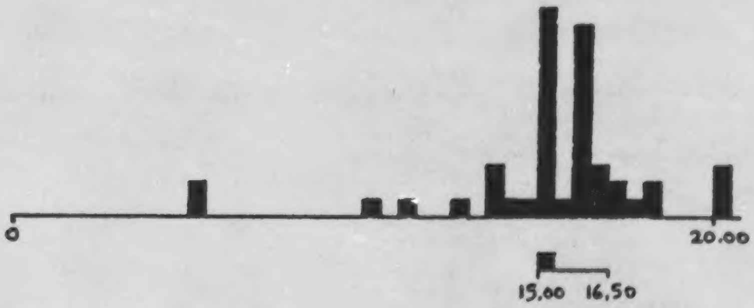
Acceptable Interval 15.00-16.50

Screening Data:

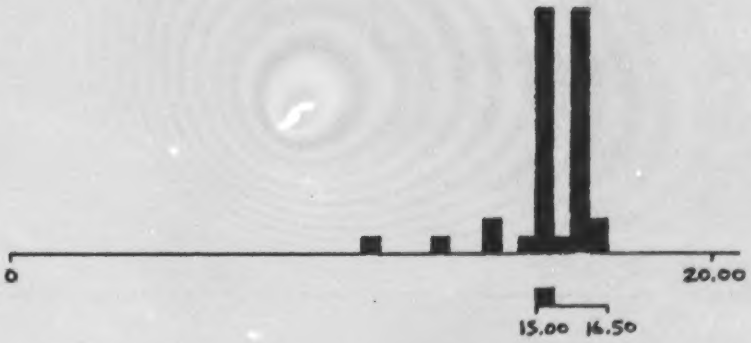
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	27	20	24	37	42	40	39	39	39	56	39	48
Discrimination Index	.47	.43	.46	.45	.38	.39	.35	.36	.34	.30	.49	.40

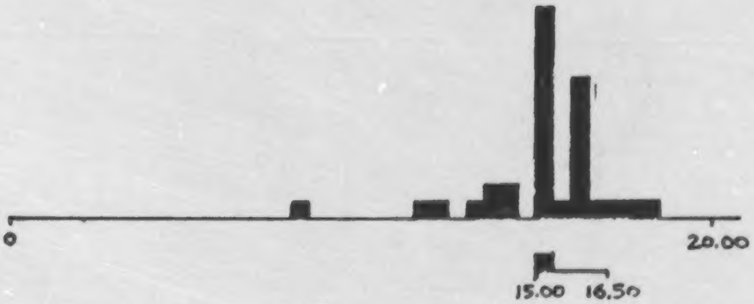
APPL 26



Grades 7,8
n=45 nr=0



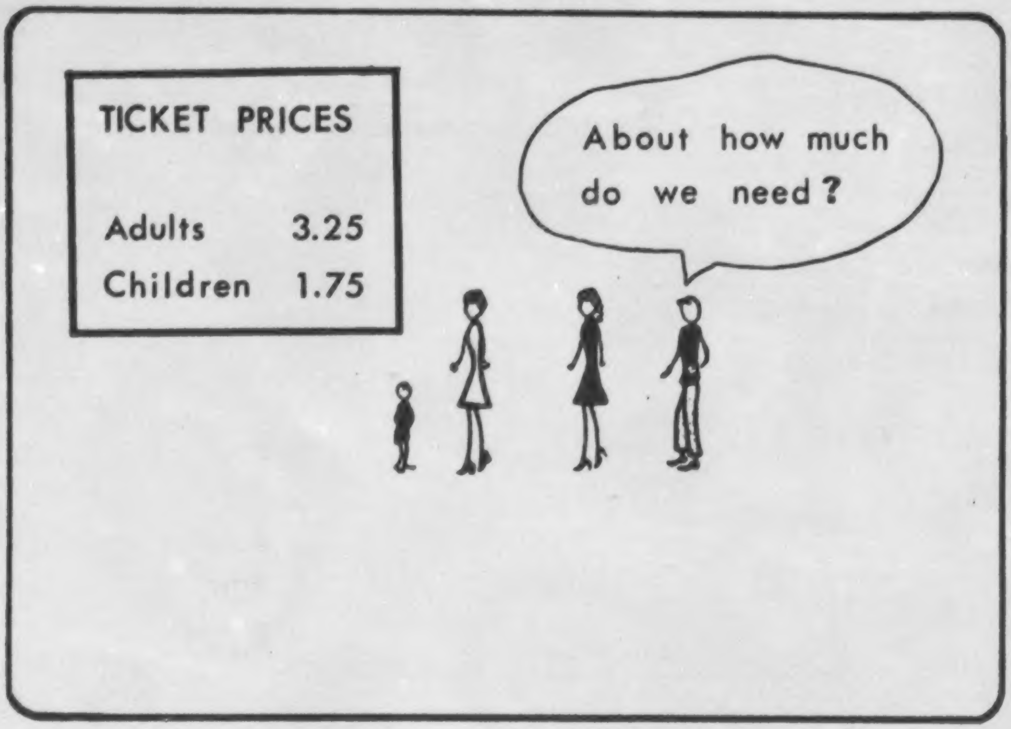
Grades 9,10
n=36 nr=0



Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 27


TICKET PRICES
Adults 3.25
Children 1.75
About how much do we need?

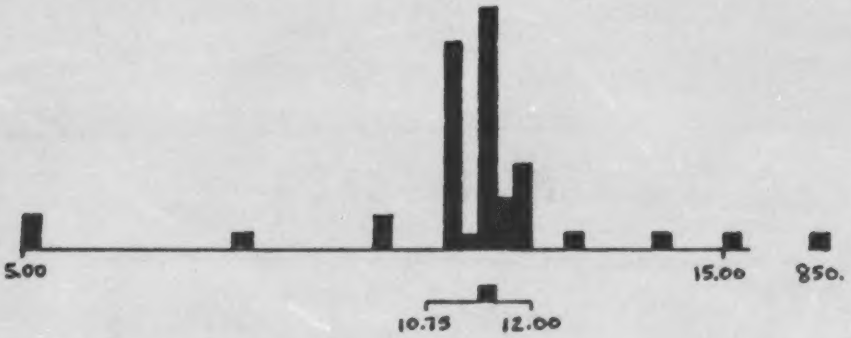
Acceptable Interval 10.75-12.00

Screening Data:

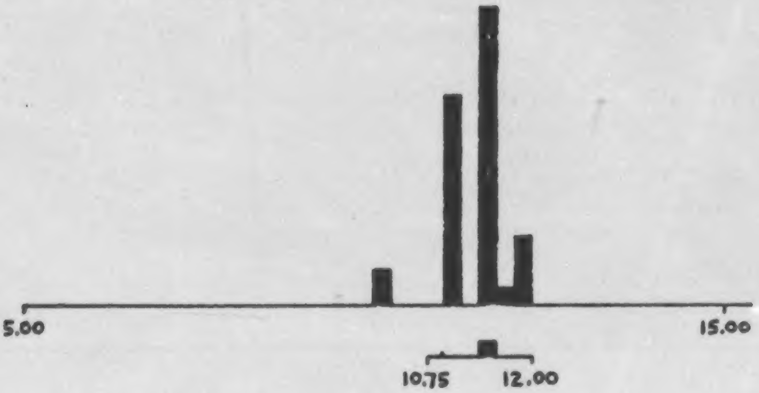
Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
Percent Correct	52	44	48	73	71	72	63	71	66	86	76	81
Discrimination Index	.49	.49	.49	.29	.40	.35	.37	.36	.35	.63	.29	.47

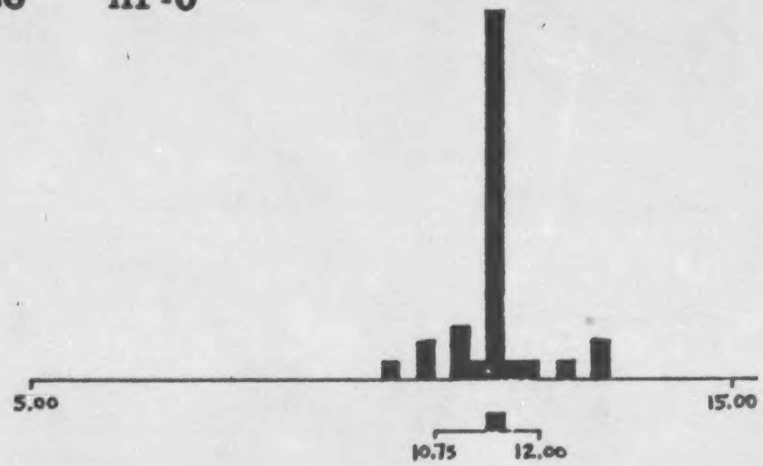
APPL 27



Grades 7,8
n=45 nr=1



Grades 9,10
n=36 nr=0



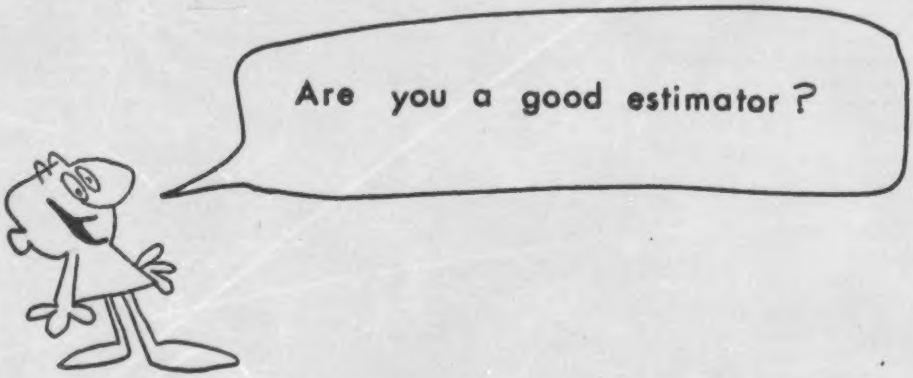
Grades 11,12
n=33 nr=0

Note: ■ represents one response

Exercise 28

CIRCLE ONE:

yes no not sure


Are you a good estimator?

Acceptable Interval ____________

Screening Data:

Time allowed: 12 sec. Time allowed: 10 sec.

	Grade 7-8			Grade 9-10			Grade 11-12			Adult		
	M (222)	F (209)	T (431)	M (154)	F (205)	T (359)	M (165)	F (126)	T (291)	M (57)	F (49)	T (106)
yes	14	5	10	20	4	11	19	6	14	49	12	32
no	29	39	34	23	62	45	25	55	38	21	59	39
not sure	57	56	57	55	34	43	53	38	47	25	29	26

INTERVIEW RESULTS

Introduction

A variety of different specific strategies was demonstrated during the interview. Due to the uniqueness of some strategies and the sketchy nature of others, it would not be productive to provide an encyclopedic listing of them. However, several general processes were observed with such regularity that they deserve to be clearly identified and described. These processes were intertwined with many specific strategies and appeared to be the reason for the selection of one technique over another. These processes also influenced the accuracy of the final estimate and reflected the personal preference of each subject. It is hoped that the characterizations of these processes will provide a framework for organizing some of the strategies used by good estimators. They are presented here to serve as an advanced organizer for assimilating all of the interview data reported throughout this section. Hopefully this organization will help readers better understand and interpret findings from the interviews.

Key Processes

Three key processes were identified that seem closely associated with good estimation skills. Each is a high level cognitive process which is difficult to operationally define. Each characterization is accompanied by an excerpt from an interview that illustrates the process in action.

A few words of caution in interpreting these processes and their characterizations:

1. The exact manifestation of these processes varied among the subjects interviewed. Therefore, the characterization reflects common actions gleaned from many responses.
2. It would be ideal if these general processes were mutually exclusive and allowed assignment of each response to a unique process. In reality it was often found that a single response contained evidence of several general processes. Therefore, the specific examples offered to illustrate a general process might also have been used to demonstrate another general process.
3. All of these processes were not demonstrated by all of the good estimators. The interview data in Table 12 and the specific responses that are reported show how frequently some of these processes were used.

Characterization of the three key processes, translation, reformulation, and compensation, together with an illustrative example of each drawn from the interview data follow.

TRANSLATION: Changing the equation or mathematical structure of the problem to a more mentally manageable form. This form was then used to computationally process the numerical data. Several different types of procedures were observed.

(a) processes numerical values in an order other than as stated in the problem but which are mathematically equivalent.

Example: $\frac{347 \times 6}{43}$ (Interview Exercise 4)

"It would be easiest to divide the 6 and 43 first which is about 7, so 347/7 is about 50." (9th grader)

(b) changes numbers to reflect an equivalent equation to accomodate particular computational preferences and strengths of the individual.

Example: 31 x 68 x 296 (Interview Exercise 2)

"I'll use 30, 70, and 300. To multiply 30 x 300, I'll change it to 3 x 3000, that way all of the zeros are on one number. Then, 9000 x 70 can be changed to 90,000 x 7 or 630,000." (12th grader)

(c) changes operations stated in problem to form an equivalent equation.

Example: (Interview Exercise 1)

$$\begin{array}{r} 87\ 419 \\ 92\ 765 \\ 90\ 045 \\ 81\ 974 \\ +\ \underline{98\ 102} \end{array}$$

"All of the numbers are close to 90,000 so it would be 90,000 x 5 or about 450,000." (9th grader)

REFORMULATION: Changing the numerical data into a more mentally manageable form. This phenomenon, which left the structure of the problem intact, was observed in several ways:

(a) front-end use of numbers

--working with one or more of the left front digits.

Example:
```
  87 419   (Interview Exercise 1)
  92 765
  90 045
  81 974
+ 98 102
```

"Add the first (front-end) digit and it gives 44 or 45 so the sum is 450,000." (9th grader)

--rounding to the nearest multiple of five, ten, hundred, etc.

Example: $8\,127 \overline{)474\,257}$ (Interview Exercise 3)

"I just rounded it (474,257) up to 480,000 and knocking that (8,127) down to 8,000. It has to be about 60." (10th grader)

Reformulation (cont.)

(b) substitution of numbers

--using a compatable number relatively close to the original number for purposes of easily operating on other data in the problem.

Example: $\frac{347 \times 6}{43}$ (Interview Exercise 4)

"I looked for nice numbers or multiples to round to - 347 to 350, 43 to 42 so you have $\frac{350 \times 6}{42}$ and cancel 6 and 42 which gives 350/7 or 50." (12th grader)

--using an equivalent or approximately equivalent form of the number, i.e., changing a fraction to a decimal or percent.

Example: The 1979 Superbowl netted $21 319 908 to be equally divided among the 26 NFL teams. About how much does each team receive? (Interview Exercise 10)

"Round dividend to 20 million and divisor to 25 then change this division problem to a fraction - 20/25 or 4/5. So each team receives 4/5 of one million or 800,000." (10th grader)

COMPENSATION: Adjustments made to reflect numerical variation that came about as a result of translation and/or reformulation of the problem. These adjustments were typically a function of the amount of time available to make a response, but were also influenced by the manageability of the numerical data, context of the problem, and the individual's tolerance for error.

This phenomenon manifested itself at two distinct stages in formulating the estimate:

intermediate compensation - adjustments that are made during intermediate stages of mental computation. These adjustments often take the form of trade-offs and are usually associated with identifiable stages of the problem.

Example: Here are 3 estimates for the total attendance for the past 6 Superbowl games:

	YEAR	ATTENDANCE
1 000 000	1974	73 655
600 000	1975	86 421
550 000	1976	91 943
	1977	96 509
Which is the	1978	93 421
best estimate?	1979	106 409

(Interview Exercise 9)

"I rounded all to 100,000 except 73,000. I dropped this one to make up for rounding others up. The numbers are so close they just make up for each other." (9th grader)

Compensation (cont.)

final compensation - an adjustment made at the end of all mental computation reflecting an awareness of the relationship of the estimate to the exact answer. Thus, an amount is added on or taken off to adjust the initial estimate.

Example: The 1979 Superbowl netted $21 319 908 to be equally divided among the 26 NFL teams. About how much does each team receive? (Interview Exercise 10)

"Round to 26 million divided by 26 teams. That's 1 million apiece, but it has to be less because of my rounding procedure, say $850,000 each." (9th grader)

Regardless of whether intermediate or final compensation was performed, an important issue centers on the size of the compensation that was determined - that is, what adjustment should be made and how was it decided? Interestingly, two different rationales for determining the amount of compensation were observed:

--adjustments which reflect specific identifiable computational schemes designed to more closely approach the exact answer.

Example: $1\frac{7}{8} \times 1.19 \times 4$ (Interview Exercise 5)

"Round to 4 x 1.2 x 2 which gives 4.8 x 2 or 9.6, but this is about 1/8 too high because of rounding 1 7/8 up to 2. So knock off 1/8 of 9.6 or 1.2, that gives 8.4." (7th grader)

Compensation (cont.)

--an intuitive feeling not necessarily associated with particular computational stages of the problem. It was often characterized by students' inability to verbalize and/or describe a specific rationale for the adjustment.

Example: $\frac{347 \times 6}{43}$ (Interview Exercise 4)

"347/43 is about 9 so 9 x 6 is 54 but it must be less because 347/43 is below 9; so I'll take off some. That leaves... I'll say 50." (7th grader)

Front-End Strategy

A number of strategies were intertwined with these key processes and are reported in the presentation of the interview data. Some of these strategies are well-known, while others are less widely recognized. One of them, a front-end strategy, was observed in varied forms and in many situations in the interviews. It has long been used by estimators, but only recently has it been identified and discussed (Trafton, 1978). In order to insure a more common understanding of this powerful, important, and frequently observed strategy, a characterization will be offered.

Here is an outline of four forms of the front-end strategy observed during the interviews.

Rounding

Subjects first rounded all or part of the numbers involved before operating, then

- operated with rounded numbers using the same number of digits (Round-SND), e.g. 4792 ⟶ 5000 or 4792 ⟶ 4800.
- operated with an extracted portion of rounded numbers (Round-EXT), e.g. 4792 ⟶ 5000 ⟶ 5 or 4792 ⟶ 4800 ⟶ 48.

Truncation

Subjects first truncated the numbers involved, then:

- replaced the right-hand digits with zeros followed by operating on these revised numbers using same

number of digits (Truncate-SND), e.g.
4792 ⟶ 4 ⟶ 4000 or 4792 ⟶ 47 ⟶ 4700.

- operated on extracted front-end digits (Truncate-EXT), e.g. 4792 ⟶ 4 or 4792 ⟶ 47.

To better understand the various uses of the front-end strategy, consider estimating the sum of this problem:

$$\begin{array}{r} 4782 \\ 5430 \\ +\ \underline{6452} \end{array}$$

A verbalization of the different applications of the front-end strategy outlined above are offered here:

Round-SND: 5000 + 5000 + 6000 is 16,000

Round-EXT: 5 + 5 + 6 is 16, so estimate is 16,000

Truncate-SND: 4000 + 5000 + 6000 is 15,000

Truncate-EXT: 4 + 5 + 6 is 15, so estimate is 15,000

The common thread among each of these forms or applications of this strategy is the focus on the front-end digits of the numbers involved.

As illustrated by Round-SND and Round-EXT, a front-end strategy may use rounding. Truncation, deleting all digits to the right of those being used by either extraction of the front-end digits or replacing the right-most digits with zeros, is also appropriate at times as shown in the latter two applications. Thus, in a front-end strategy the left-most digits or a rounded form of them is used in the mental computation. In some cases, as in

Truncate-EXT, the front-end digits are extracted from each number, operated on, then the appropriate number of zeros added. In other instances, as in Truncate-SND, the front-end digits are extracted while the right-most digits are replaced with zeros. This use of truncation was explained by many students as allowing the problem to be "easier to handle".

Each of these forms has advantages depending on the particular numerical situation. While the rounding strategies may, in certain instances, result in a more accurate estimate, the truncate strategies enable the user to visualize the digits being operated.

The exact number of digits used is a function of several variables such as the size of the numbers, the operations involved, the amount of time available to formulate an estimate and the accuracy of the estimate that is desired. Furthermore, this strategy can be further characterized as front-end rounding or front-end truncation, depending upon the procedures used in a particular problem.

A variety of rounding methods was exhibited in conjunction with this strategy including rounding to multiples of ten, rounding to mentally manageable numbers, and rounding to multiples which produce compatible numbers with which to operate. Compatible numbers are those groups of numbers which, when used in conjunction, are easily operated on, e.g. the first and third addend of 2314 + 812 + 1737 might produce an estimate such as: (2314 + 1737) + 812 $\approx$ 4000 + 812 or 4812. Partial rounding, that is, rounding of some but not all of the numbers involved, was employed and often explained as a more accurate use of rounding for estimation.

Several of the key processes described earlier were observed as subjects used the front-end strategy. For example, grouping of compatible numbers was commonly observed when the front-end strategy was employed, especially with addition. Initial grouping and operating on a subset of the numbers in a problem often alleviated the need for compensation as well as made the problem more mentally manageable. A similar technique used in conjunction with the front-end strategy was operating on numbers out of their presecribed order.

In summary, the general front-end strategy is a very versatile and powerful technique. It was consistently employed by good estimators and was not recognized as having been taught in their school mathematics program. The flexibility of the front-end strategy enabled users to estimate more quickly as well as made the task of estimating less computationally taxing. The front-end strategy was observed with all four operations as well as with different types (decimals or whole) and magnitudes of numbers.

Common Strategies and Techniques Observed in the Interviews

What are the strategies used by good computational estimators? The strategies used by good estimators at all levels seem to have resulted from an interaction of several complex variables. These include the experiential background of the individual, the mathematical operation being performed, the size of the numbers, and the relationship of the numbers within a given problem. The interview battery was not designed to study the effects of all of these variables nor was its scope sufficient to reach meaningful generalizations with regard to them. Ascertaining variations in computational estimation procedures was the overriding objective in constructing the battery, and at best, the researchers identified the strategies verbalized by the subjects as they worked each problem. Nevertheless, an analysis of these interview data was productive in revealing specific strategies as well as some similarities in the processes these good estimators used when confronted with certain problems.

Since the response to each interview exercise was eventually associated with an estimation strategy, it was important to have agreement on this categorization of response. In other words, the reliability of interview scoring was of crucial importance. Initial analysis of the data from the 15 straight computation and application exercises revealed many identifiable strategies. Whenever a particular strategy appeared more than once, it was identified and defined as clearly as possible. This process provided the classification system for the interview results reported in Table 12. Each interview was coded by this classification system. A measure of the realiability of these results was obtainted by randomly choosing a subject and having each of the four researchers independently code the responses from the audio tape interview. The four researchers agreed unanimously on 87 percent of the responses and at least three of them agreed on 95 percent of them. This suggests a high interrater agreement, and further documents that the strategies as defined were generally agreed upon by these researchers.

Table 12 summarizes these interview data for each problem and provides a basis for the discussion of the most prevalent strategies. The description of strategies used with each exercise contains a frequency of strategies used by the 59 subjects who were interviewed. The total number of responses for each exercise varies because of other strategies that were used without sufficient frequency for categorization. Each example has two rows of frequencies listed to its right. The second row indicates the frequency of subjects in the respective sub-groups who used compensation with the described strategy. The description of Exercise 1 of Table 12 indicated that four different general strategies were observed. Several different forms of these general strategies were also identified, each of which depended upon the selection of numbers used by the subject in making their estimate. For example, of the eleven subjects who employed an average to 100,000 strategy, nine went beyond this and used compensation to further refine their estimate. Of the four eleventh and twelfth graders in this category, two produced an answer not in the acceptable range. In addition to the frequency with which various strategies were observed, this table also provides illustrative examples of the most frequently observed strategies. These examples appear on the facing page of each problem.

Table 12. Summary of Subject Responses to Interview Exercises

Exercise 1

87 419
92 765
90 045 Acceptable Interval (430 000 to 460 000)
81 974
+ 98 102

Characterization of Estimation Strategy Used	Frequency of Subjects Using This Strategy				
		Grade			
Average - exhibits holistic view of problem by observing all or most numbers center about a particular value	Total	7,8	9,10	11,12	Adults
Average to 90 000	14	0	3	3	8
comp *	3	1	0	1	1
Average to 100 000	2	0	0	1	1_1
comp	9	1	3_3	4_2**	1_1
Truncate-EXT - operating on extracted front-end digits					
First column	1	0	0	0	1
comp	2	0	2	0	0
First two columns	2	1	0	1_1	0
comp	1	0	0	1	0
Round-SND or EXT - operating in given order on rounded numbers or extracted portion of rounded numbers					
Rounding to 1000's	6	2	3	1	0
comp	0	0	0	0	0
Rounding to 10 000's	9	4	4_1	1_1	0
comp	3	1	0	2_2	0
Round/Group - operating on rounded numbers or extracted portion of rounded numbers by grouping manageable numbers					
To 10 000's	5	1	3	1	0
comp	2	0	1	0	1
	59	11	19	16	13

* Compensation used in conjunction with this strategy.

** Indicates that 2 of 4 of the responses were not in the acceptable interval.

Exercise 1
Illustrative Subject Responses

Average numbers to about 90 000; 90 000 x 5 is 450 000 plus about 2000 for the hundreds digits which gives 452 000.

About 100 000 each; 5 x 100 000 or 500 000, then went back to see about how far each was from 100 000 and subtracted that (10, 10, 10 and 20) from 500 000 which gives about 450 000.

Added the first digits (ten thousands): 9, 8, 9, 8, and 9 to about 44 or 45 which means 450 000.

Added front-end digits (ten thousands and thousands) one column at a time starting with ten thousands.

Rounded then added two front-end digits: 87, 93, 90, 82, 98 -- about 440 000.

Rounded to nearest 10 000; three 90's, and two more 90's (averages 100 and 80). That's five 90's or 450 000. "With numbers that big, ten thousands is good enough."

Rounded first three to 90 so 3 x 90 is 270, 270 + 80 is 350 plus 100 is 450 000.

Table 12 (cont.)

Exercise 2 31 x 68 x 296 Acceptable Interval (600 000-634 000)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Round-SND - rounded each number to a ten or hundred and multiplied, using the same number of digits		Total	7,8	9,10	11,12	Adult
In order (30 x 70) x 300		19	5_2	9_3	2	3_1
	comp	8	2_2	2	3_1	1_1
Largest first (300 x 70) x 30		7	0	1	3_1	3_1
	comp	1	0	1_1	0	0
Easy numbers first (30 x 300) x 70 or (300 x 30) x 60		3	0	3_1	0	0
	comp	1	0	0	1	0
Round-EXT - rounded, extracted front-end digits, multiplied then added zeros						
3 x 7 x 3 (then count zeros) or (3 x 3) x 7		14	2	1	5_2	6_1
	comp	1	1	0	0	0
		54	10	17	14	13

Other Strategies Used:

(30 x 60) x 300

(30 x 270) x 70

68 x 10 x 3 x 100 x 3

(30 x 70) x 296

Exercise 2
Illustrative Subject Responses

30 x 70 is 2100 and 2100 x 300 is 630 000.

300 x 70 is 21 000 ("I always do the harder (bigger) ones first.") and 21 000 x 30 is 630 000--". . . since I rounded up and down, I didn't have to knock off any (compensate)."

30 x 300 is 9000 ("I worked with easiest numbers (30 and 300) first.") and 9000 x 70 is 630 000.

30 x 70 x 300 is 21 x 3 which is 63 then added on 4 zeros.

Table 12 (cont.)

Exercise 3 $8\,127\,\overline{)474\,257}$ Acceptable Interval (50 - 62)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
		Total	7,8	9,10	11,12	Adult
Truncate-EXT - extracted some of leading digits						
$8\,\overline{)47}$		4	4_4	0	0	0
	comp	6	0	4_1	2_2	0
$8\,\overline{)474}$		1	0	0	0	1
	comp	4	0	1	2	1
Round-SND - rounded numbers then operated using same number of digits						
$8000\,\overline{)480\,000}$ or $8000\,\overline{)400\,000}$		17	4_1	8_3	3_3	2_1
	comp	4	1	1	0	2_1
$8000\,\overline{)500\,000}$		5	0	0	5_2	0
	comp	0	0	0	0	0
Round-EXT - rounded numbers then operated using extracted front-end digits						
$8\,\overline{)480}$		2	1	0	0	1
	comp	0	0	0	0	0
$8\,\overline{)48}$ or $8\,\overline{)50}$		6	0	2_1	1	3
	comp	2	0	1_1	0	1
Partial Rounding - rounded one of numbers, using same number of digits						
$8000\,\overline{)474\,257}$ (algorithmic process)		1	0	0	1_1	0
	comp	2	0	1	0	1
Multiplication - used related multiplication sentence						
8000 x __ = 480 000		0	0	0	0	0
	comp	4	1_1	0	2	1
		58	11	18	16,	13

Other Strategies Used:

$10\,000\,\overline{)500\,000}$

Exercise 3
Illustrative Subject Responses

Divided 8 into 47 - about 5, almost 6, so estimate is about 58.

Divided 8 into 474 - gives 59 ". . . other digits (that were truncated) won't affect quotient too much."

Rounded 474 257 up to 480 000 and 8127 down to 8000. "I looked for easy multiples. It's probably going to be a little below 60 000."

Rounded to 8000 and 500 000 so "It would be more than 60, but a lot less than 70, so about 63."

Rounded to 480 000 divided by 8000 which is equivalent to 480 divided by 8 or about 60.

Rounded to 480 000 divided by 8000. "Change this to 48 divided by 8 or 6 then count zeros--4 (number of zeros in 480 000) minus 3 (number of zeros in 8000) is 1 so 6 with 1 zero or 60 is the estimate."

Partially rounded to $8000\overline{)474\ 257}$ and go through mentally algorithmic process to obtain quotient of 59.

Rounded to 8000 divided into 500 000. Used multiplication, missing factor strategy. "If I multiply it (8000) by 60 it would be 480 000 so that (60) is close enough."

Table 12 (cont.)

Exercise 4 $\frac{347 \times 6}{43}$ **Acceptable Interval (42 - 60)**

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Round-SND (in order) - operating in order on rounded numbers, using same number of digits		Total	7,8	9,10	11,12	Adult
$(300 \times 6) \div 40$		5	0	2	2	1
	comp	2	1_1	0	1_1	0
$(350 \times 6) \div 50$		4	1	0	3_1	0
	comp	1	0	1	0	0
$(350 \times 6) \div 40$		12	3_1	5_1	1_1	3_1
	comp	14	3_1	5	3_1	3
Round (out of order) - operating on rounded numbers in an order other than in a left to right manner						
$\frac{6}{43}$ is about $\frac{1}{7} \times 350$		3	0	1	1	1
	comp	3	0	0	1	2
Round to $\frac{320}{40} \times 6 = 8 \times 6$		8	0	3	3	2
or $\frac{360}{40} \times 6 = 9 \times 6$	comp	1	0	0	0	1
or $\frac{400}{40} \times 6 = 10 \times 6$						
or $\frac{350}{50} \times 6 = 7 \times 6$						
		53	8	17	15	13

Other Rounding Strategies Used:

$(300 \times 5) \div 50$

$(300 \times 6) \div 50$

$(347 \times 6) \cong 2042 \div 43 \cong 50$ or 55

$(347 \times 6) \cong 2002 \div 40 \cong 50$

$(350 \times 5) \div 50$

$(35 \times 6) \div 4$

Exercise 4
Illustrative Subject Responses

"Changed 347 to 300 x 6 and then added on a little bit to get 2000. . . then divided 2000 by 40 which gives 50."

"I would go 347 and take it to 350 x 6 and then 6 x 300 is 1800 and 6 x 50 is 300, so you would have 2100. I'd round the 43 to 40, so 2100 divided by 40; I'd take how many times 4 would go into 21 which would be 5. . . I'd say about 52."

Rounded to 350 x 6 which is 2100, then change this to 2000 and divided by 40 or about 50.

"Looked for multiples to round to: 347 to 350, 43 to 42 so you have $\frac{350 \times 6}{42}$ and cancel 6 and 42 which gives $\frac{350}{7}$ or 50."

"Divide first--say it's about 320 ÷ 40 or 8 then times 6 which is 48."

Table 12 (cont.)

Exercise 5 $1\frac{7}{8} \times 1.19 \times 4$ Acceptable Interval (8 - 10)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy Total	7,8	9,10	11,12	Adult
Round to whole numbers -						
2 x 1 x 4		31	6	8	8_1	9
	comp	3	1	0	1	1
Round to decimals -						
In order: (2 x 1.2) x 4		1	0	0	1	0
	comp	1	0	1	0	0
Different order: (1.2 x 4) x 2 or (2 x 4) x 1.2 or (4 x 1.19) x 2		4	1	1	1	1
	comp	7	1	5	1	0
Round to fractions -						
$1\frac{7}{8} \times 1\frac{1}{5} \times 4$		1	0	0	1	0
	comp	1	0	1	0	0
Round to numbers using both fractions and decimals -						
$1\frac{7}{8} \times 4 \times 1.19$		0	0	0	0	0
	comp	1	1_1	0	0	0
$4 \times 1.2 \times 1\frac{7}{8}$		2	0	0	1	1
	comp	3	1	1	1	0
		55	11	17	15	12

Other Strategies Used:

(1.1 x 4) x 2

$(1.19 \times 4) \times 1.6 \approx 4.7 \times 1.6 = 4 \times 1 +$ some

$\frac{15}{8} \times 4 \times 1^{+}$

Exercise 5
Illustrative Subject Responses

Rounded 1 7/8 up to 2, 1.19 down to 1 and multiplied by 4. "I can get away with that alot better because when you round it up that way and round it down that way it will come out pretty close to 8. I took care of the harder ones first."

Rounded 1 7/8 to 2 then multiplied by 1.2 then by 4, which gives 9.6.

Changed 1.19 to 1.2. 1.2 x 4 is 4.8 times about 2 is 9.6, then compensated. "It (1 7/8) is not quite 2 so I'll take some off so the estimate is about 9."

1.19 is about 6/5 so $\frac{6}{5} \times \frac{15}{8} \times 4$ is $\frac{6}{5} \times 4$ which is $\frac{24}{5}$ then times $\frac{15}{8}$ or about 15 fourths (incorrect). "Oh, that's not right, all the numbers are above 1 and 4." (Student recognized error and obtained a correct estimate using same strategy.)

1 7/8 is $\frac{15}{8} \times 4$ or $\frac{15}{4}$ then this times 1^{+} is about 5.

4 times 1.2 is 4.8 then change 1 7/8 to $\frac{15}{8} \times \frac{4.8}{1}$ is 15 x .6 or 9.

Table 12 (cont.)

Exercise 6	What is the area of the rectangle? (28 by 47)	Acceptable interval (1200 - 1500)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy	Grade			
Round-SND -		Total	7,8	9,10	11,12	Adult
Changed one factor 30 x 47 or 50 x 28		1	0	1	0	0
	comp	2	0	2	0	0
Changed both factors						
30 x 50		20	8	5	3	4
	comp	27	2	10$_1$	9	6
30 x 45		4	0	0	2	2
	comp	1	0	0	1	0
Distributive Principle - use of mental computation using distributive principle						
(8 x 47) + (20 x 47)		6	1	0	1	0
	comp	7	0	1	0	0
		58	11	19	16	12

Exercise 6
Illustrative Subject Responses

Round one factor to the nearest multiple of 10 - 47 x 30 is 1410.

Round 28 to 30 and 47 to 50. Multiplied 30 x 50 which is 1500, compensated downward. "Since I rounded both of them up, I'd probably drop the estimate back to 1450."

Round to 45 x 30 = 1350. "This is rounding up one factor and down on the other--this is what you want to do, if possible."

Multiplied 8 x 47 (376) and added that to 20 x 47 (940) to arrive at about 1300.

Table 12 (cont.)

Exercise 7 30% of 106 409 Acceptable Interval (30 000-36 000)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Round-SND - operating in given order on rounded numbers, using same number of digits		Total	7,8	9,10	11,12	Adult
Using decimals - verbalized changing 30% to .3:						
.3 x 106 000		0	0	0	0	0
	comp	2	0	0	2_1	0
.3 x 100 000		6	2	1	2	1
	comp	2	0	1	1	0
Using fractions - verbalized changing 30% to an approximately equivalent form, 1/3:						
$\frac{1}{3}$ of 105 000 or 106 000		8	2	3_1	2_1	1
	comp	4	1	3	0	0
$\frac{1}{3}$ of 100 000		1	0	1	0	0
	comp	2	0	0	2	0
Using percent - verbalized use of 30% without mention of an equivalent form:						
30% of 106 000		2	0	0	1	1
	comp	4	1	1	1	1
30% of 100 000		7	1	3	0	3_1
	comp	4	2	1	0	1
10% of 106 000 x 3 or 10% of 100 000 x 3		5	0	2	1	2
	comp	2	1	0	0	1
Round-EXT -						
$\frac{1}{3}$ of 105 or 106 or 1064		6	1	1	3_1	1
	comp	0	0	0	0	0
		55	11	17	15	12

Other Strategies Used:

30% of 110 000

$3\overline{)106\ 409}$ (algorithmic procedure)

Exercise 7
Illustrative Subject Responses

Took 3 times 106 000 or 318 000. Placed decimal by determining that a reasonable answer would be in the 30 000's. "So the decimal must go in to make 31 800."

Changed 30% to .3, so 100 000 x .3 is 30 000.

Changed 30% to about one third, 1/3 of 100 000 is about 33 000 and 1/3 of 6000 is 2000, added these for 35 000.

Reasoned that 30% is about 1/3, "Take a third and then subtract a little, that leaves about 30 000."

"30% of 100 000 is 30 000 and 30% of 6409 is about 1800. Tacked on a little more for 409--about 32 000."

30% of 100 000 is 30 000--about 32 000 ". . . because of rounding down by 6000."

"Take 10% of that (106 000) and then multiply it by 3. That gives 31 800."

Changed 30% to about 1/3. 1/3 of 106 is 32, so 32 000.

Table 12 (cont.)

Exercise 8 8483 hotdogs @ $.60 Acceptable Interval (4200-5400)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Round-SND - operating in given order on rounded numbers, using same number of digits		Total	7,8	9,10	11,12	Adult
Using decimals:						
8000 x .60		6	0	4	1	1
	comp	11	2	5_1	2	2
8500 x .6 or 8400 x .6		7	3_1	3	1	0
	comp	2	0	0	1	1
8500 x .5		0	0	0	0	0
	comp	2	2	0	0	0
Using fractions:						
$\frac{1}{2}$ of 8000 + $\frac{1}{10}$ of 8000		3	0	1	1	1
	comp	0	0	0	0	0
$\frac{1}{2}$ of 8000, 8400, or 8500		2	0	0	1	1_1
	comp	10	2	3_2	1	4
$\frac{3}{5}$ of 8500		2	0	1	1_1	0
	comp	0	0	0	0	0
Using whole numbers only:						
8000 x 6 ÷ 10 or 8000 ÷ 10 x 6		1	0	0	1	0
	comp	1	1	0	0	0
Round-EXT - operating on extracted front-end portion of numbers						
Using decimals:						
.6 x 84 or .6 x 85		3	0	0	3	0
	comp	0	0	0	0	0
		50	10	17	13	10

Other Strategies Used:

9000 x .6

8483 x 6 (mental computation)

At $1.00 per hotdog, it would be $8 483, then compensate downward.

1/2 of 8483 plus 10% of 8483.

2/3 of 8400

Exercise 8
Illustrative Subject Responses

Used 8000 as opposed to 8500 ("Thousands are easier to multiply.") and multiplied by 60 cents--$4800, "A little more than that."

Rounded to 8500 x .6, then 8500 x 6 = 51000 and moved the decimal to the left one place to yield $5100.

Round the number up to 8500 and the price to 50¢ each, so 8500 x .5 gives 4250. It should be a little more because of knocking off the dime--say it's $5000.

Split $.60 into .5 and .1, took 1/2 of 8000 and added 1/10 of 8000 to give 4000 + 800 or 4800.

Reasoned that 60¢ is about 50¢ or half a dollar so 1/2 of 8483 is about 4200, probably a little more than this because of rounding down (8483 to 8400) and because of the extra 10¢ not used.

Changed 8483 to 8500 and 60¢ to 3/5 of a dollar, then divided 8500 by 5 and multiplied by 3 -- 5100.

Changed 8483 x .60 to 8000 x 6 or 48000, then divided by 10 by dropping off a zero.

Computed .6 x 84 which is about 50 then moved decimal to make a reasonable answer, $5000.

Table 12 (cont.)

Exercise 9	Total attendance?	Acceptable Choice
	73 655 86 421 91 943 96 509 93 421 106 409	550 000

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Average - exhibits holistic view of problem by observing all or most numbers center about a particular value		Total	7,8	9,10	11,12	Adult
Average to 90 000		3	0	1	2	0
	comp	2	0	1	0	1
Average to 100 000		2	0	0	1_1	1_1
	comp	19	5	5	4	5
Truncate-EXT - operating on extracted front-end digits						
First one or two columns		1	0	0	1	0
	comp	2	0	1	1	0
Round-SND - operating in order on rounded numbers, using same number of digits						
To 1000's		2	1	0	1_1	0
	comp	0	0	0	0	0
To 10 000's		6	2	3_1	1	0
	comp	2	0	2_1	0	0
With grouping - grouped selected rounded numbers together with other manageable rounded numbers to aid in computation		9	2	2_1	2	3
	comp	3	1	0	1	1
Elimination - narrowed choice by observing the size and number of addends, eliminating unreasonable choices		7	0	3	2	2
	comp	0	0	0	0	0
		45	11	18	16	13

Exercise 9
Illustrative Subject Responses

Reasoned that each number is about 90 000 so 6 times this is about 540 000.

Averaged each to 100 000 so 6 x 100 000 is 600 000 and each averages about 10 below 100 000 so subtracted off about 10 000 for each -- 550 000.

Reasoned that 106 and 93 would be 200 and 91 and 96 would make 387; 387 and 86 would be 483 and 73 more is around 560, so estimate is 560 000.

Rounded to nearest 1000 then mentally added in order 74 + 86 + 92 + 97 + 93 + 105.

Rounded to 70, 90, 90, 100, 90 and the 106 to 90 ("because of rounding the other ones up")--adding these gives about 550 000.

Added top two numbers (70 and 90) and bottom two numbers (100 and 90) to give 350 000. Then 200 000 for the middle two numbers so that makes 550 000.

Reasoned that: "One million is out because there were only six numbers and all but one was less than 100 000. So this means 600 000 is out too, it's gotta be less than that--say 550 000.

Table 12 (cont.)

Exercise 10 The 1979 Superbowl netted $21 319 908 to be equally divided among the 26 NFL teams. About how much does each team receive?

Acceptable Interval (700 000 to 950 000)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
		Total	7,8	9,10	11,12	Adult
Round to Manageable Numbers - selected rounding schemes producing divisor and dividend which were evenly divisible.						
$25\,\overline{)20\ 000\ 000}$ or		6	2	1	1	2
$30\,\overline{)21\ 000\ 000}$	comp	5	0	0	4	1
$26\,\overline{)26\ 000\ 000}$ or		1	0	1_1	0	0
$21\,\overline{)21\ 000\ 000}$ or	comp	25	6_1	10	3	6
$20\,\overline{)20\ 000\ 000}$						
Truncate-SND - operating on front-end portion of number, using same number of digits						
$26\,\overline{)21\ 000\ 000}$ or		4	1	3	0	0
$26\,\overline{)21\ 300\ 000}$	comp	8	1	2	5	0
Ratio Reasoning - changed numbers to a ratio form verbalizing their relation to each other then converting this relationship to a numerical estimate						
$\frac{21}{26}$ or $\frac{20}{25}$		4	1	1	0	2
	comp	1	0	0	1	0
		54	11	18	14	11

Other Strategies Used:

$30\,\overline{)21\ 319\ 908}$ (algorithmic process)

Round to $30\,\overline{)20\ 000\ 000}$

Exercise 10
Illustrative Subject Responses

Rounded to $20 000 000 and 25 teams, then divided to yield $800 000 each ". . . but need to add a little more due to rounding procedure used . . . $830 000 is a good 'guesstimate'."

Reasoned that 26 teams at $1 million would be $26 million or about 5 million more than what was taken in, so 5 million divided by 26 teams is about 200 000 off. So each team got about $800 000.

21 divided by 26 gives an 8 in the hundred thousands place, then added zeros to fill it out.

26 $\overline{)21}$ is equivalent to 21/26 or about 20/25 = 4/5 which is .8. Verbalized that to make a reasonable answer, must move decimal to 800 000.

Table 12 (cont.)

Exercise 11 Three people have dinner.
(See Appendix 4 for menu)
About how much will be needed to pay the bill?

Acceptable Interval
($8 - $11)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
Round-SND (in order) -		Total	7,8	9,10	11,12	Adult
To nearest 50¢		13	0	7	3	3
	comp	4	0	1	1	2
To nearest 10¢ (dime)		12	5	3	2	2
	comp	3	0	0	1	2
Round-SND (with grouping) -						
To nearest $1.00		5	0	0	2	3
	comp	5	2	2	1	0
To nearest 50¢		7	1	3	2	1
	comp	1	0	0	1	0
To nearest 25¢		1	0	0	1	0
	comp	0	0	0	0	0
To nearest 10¢ and/or 5¢		4	3	1	0	0
	comp	1	0	0	1	0
		56	11	17	15	13

Other Strategies Used:

Mentally computed exact amount.

Exercise 11
Illustrative Subject Responses

Rounded each to the nearest half-dollar amount ($3.00 + $2.50 + $1.50 + $0.50 + $1.00 + $0.50) then added in order.

Scanned menu, rounded to nearest dime amounts and kept a cumulative total. Operated on numbers in order given.

Grouped prices by dollar amounts, then added these dollar amounts. Compensated by rounding down when felt subtotal was overestimate.

Rounded numbers to nearest 50¢ amount then searched for and operated on compatable numbers.

Rounded to nearest 25¢ amount, then searched for compatable numbers to obtain subtotals.

Rounded to 5¢ and 10¢ amounts (depending on which is most compatable with present subtotal). Jumped around looking for "neat" (compatable) numbers to work with.

Table 12 (cont.)

Exercise 12 The Thompson's dinner bill totaled $28.75. Mr. Thompson wants to leave a tip of about 15%. About how much should he leave for the tip?

Acceptable Interval ($3 to $5)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy		Grade		
		Total	7,8	9,10	11,12	Adult
Use of Fractions - changed fraction to an approximately equivalent form then computed using it						
15% to 1/7		2	1	1	0	0
	comp	1	0	1	0	0
15% to 1/6		3	1	2	0	0
	comp	0	0	0	0	0
Use of Decimals - verbalized changing percent to decimal then computed with that decimal form						
.15 x $30 or .15 x $29		5	1_1	2	2	0
	comp	5	0	1	3	1
Use of Percent - verbalized use of a percent approximate to 15% without mention of converting this percent to an equivalent form						
10% of 30		0	0	0	0	0
	comp	4	1	1	1	1
15% of 30		5	1	2	1	1
	comp	2	0	1	0	1
20% of 30		0	0	0	0	0
	comp	2	1_1	0	1	0
Distributive Strategy - operation of 15% handled through two-step distributive procedure using percents, decimals or whole numbers						
10% of 28 or 30 + half of that		22	2	7	6_1	7
	comp	6	2	0	2	2
		57	10	18	16	13

Other Strategies Used:

Change 15% to 1/5 then 1/5 of $30

15/100 ≈ 5/33 so $5 for every $33, so about $4.

Reduced 15% to 3/20 then 3/20 x $30.

Exercise 12
Illustrative Subject Responses

Reasoned that 15% is about 1/7 and 1/7 of $28.75 is about $4.

Reasoned that 15 goes into 100 about 6 times so 15% is about 1/6. 1/6 of $30 is 5.

Rounded $28.75 to $30. Then 15% is same as .15 so 30 x .15 is $4.50.

Computed 10% of $30 to be $3.00. Reasoned that this was off by 5% which is less than $1.50 more -- compensated to between $4.00 and $4.25.

Rounded to 30 x 15 or 450. Placed decimal by what seemed to be reasonable answer. "Should be a little less, say $4.30."

Mentally computed 20% of $30 to be $6.00, so, "15% is lower or about $4.50".

Reasoned that: "15% of this is 10% of $28.75 plus half of that or 2.88 + 1.44 which is about $4.30, probably a little lower."

Table 12 (cont.)

Exercise 13* Which carton has more soda? Acceptable Response (COKE)

COKE	PEPSI
6 32-oz. bottles	8 16-oz. bottles

Characterization of Estimation Strategy Used	Frequency of Subjects Using This Strategy				
		Grade			
Computed and Compared Total Ounces - estimated ounces in carton and compared totals	Total	7,8	9,10	11,12	Adult
Using Rounded Numbers (6 x N and 8 x N -- N indicates rounded form of 32 and 16)	18	3	8	3	4
Using Exact Numbers (6 x 32 and 8 x 16)	15	1	4_1	6	4
Using Partial Rounding (rounded either 16 oz. or 32 oz. but not both)	5	0	1	3	1
Equivalent Forms with Comparison - set up an equivalent correspondence between brands using bottles or ounces, then compared to actual values stated in problem	13	5	3	3	2
Computed and Compared Using Other Units - converted to other units of measure then compared units					
Using Gallons or Quarts	2	1	1	0	0
Using 16 or 32 ounces	6	1	2	1	2
	59	11	19	16	13

* Compensation was neither appropriate nor observed on this exercise. Therefore, the second row which has been reporting compensation was omitted.

Exercise 13
Illustrative Subject Responses

Rounded to 6 x 30 = 180 and 8 x 15 = 120 then compared.

Took 6 x 30 = 180 and 6 x 2 = 12 which gives 192 ounces for COKE, then 8 x 10 = 80 and 8 x 6 = 48 so 128 ounces for PEPSI, therefore, COKE has more soda.

Computed 8 x 16 to be 128 ounces -- compared this to 6 x 30 (rounded downward) or 180 ounces. "Even with rounding down, more soda in COKE."

Reasoned that the COKE bottle is twice as large as PEPSI bottle so twice as many bottles of PEPSI would be needed to equal the amount of soda in COKE. Since there were less than twice as many bottles, must be less PEPSI.

The PEPSI contains 4 quarts of soda whereas the COKE contains 6 quarts of soda.

There are 8 16-ounce bottles of PEPSI. "If the same amount of PEPSI were in 32-ounce bottles you'd fill 4 of them but you have 6 32-ounce bottles of COKE so it (COKE) must contain more soda."

Table 12 (cont.)

Exercise 14 * Which soda is the cheapest? Acceptable Response (COKE)

COKE	PEPSI
6 32-oz. bottles $1.79	8 16-oz. bottles $1.29

Characterization of Estimation Strategy Used	Frequency of Subjects Using This Strategy				
		Grade			
Computed Price/Unit and Compared -	Total	7,8	9,10	11,12	Adult
Computed Exact Ounces	9	4_1	0	2	3
Estimated Ounces Using Rounded Numbers	18	2	6_1	8_1	2_1
Converted to other units and compared price/unit	7	0	5_1	0	2
Compared Ratio of Units to Ratio of Prices -					
Using a 3 to 2 ratio	12	3	4	2	3
Uncertain Decision -					
Guessed	4	1	0	1	2
I don't know	2	0	2	0	0
	52	10	17	13	12

Other Strategies Used:

50¢ more for the extra 64 oz. (of COKE). This is less than a penny an ounce--COKE cheapest.

Almost twice as much COKE at less than twice the price--COKE cheapest.

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* Compensation was neither appropriate nor observed on this exercise. Therefore, the second row which has been reporting compensation was omitted.

Exercise 14
Illustrative Subject Responses

"The COKE has 192 oz. at $1.79 so 179/192 is less than 1¢ per ounce. The PEPSI has 128 oz. at $1.29 so 129/128 is more than 1¢ per ounce."

"The COKE is 6 x 30 or 180 oz. at $1.79 so 179 ÷ 180 is less than 1¢ per ounce. The PEPSI is 8 x 15 or about 120 oz., so $1.29 ÷ 120 is more than a penny per ounce."

"The PEPSI is equivalent to 4 - 32 oz. bottles or $1.29 ÷ 4 which is about 32¢ per unit while the COKE has 6 - 32 oz. bottles or $1.79 - 6 is about 30¢ per 32 oz. unit so COKE is cheaper."

"The ratio of COKE to PEPSI is 8 to 12 or 3 to 2. So an additional half of this ($1.29) is $.64. 1.29 + .64 is about 1.93 and that is more than the COKE, therefore, price per ounce PEPSI costs more."

Verbalized that this was a difficult problem to estimate and said that big amounts usually cheapest so COKE probably best buy.

Verbalized correctly how to find solution with paper/pencil--however, said this problem contained too many digits and too many operations to handle them mentally.

Table 12 (cont.)

Exercise 15	This is a grocery store ticket which has not been totaled (see Appendix 3). Estimate the total.	Acceptable Interval ($11 to $14)

Characterization of Estimation Strategy Used		Frequency of Subjects Using This Strategy				
			Grade			
		Total	7,8	9,10	11,12	Adult
Round-SND - use rounded form of each price, operating in order						
To nearest 10		15	3	6	1	5
	comp	3	0	2	0	1
To nearest 5 or 10		5	1_1	4	0	0
	comp	6	0	1	5	0
Round/Group - used a rounded form, grouping prices by dollar amounts or amounts easily added together						
To dollar or half-dollar amounts (in order)		4	0	1	1	2
	comp	8	2	4	0	2
To dollar amounts (out of order)		3	0	0	3	0
	comp	1	1	0	0	0
To nice numbers (out of order)		5	1	0	1	3
	comp	2	0	0	2	0
Truncate-EXT - used extracted portion of numbers either in or out of order						
First two digits		2	0	0	2	0
	comp	0	0	0	0	0
		54	8	18	15	13

Other Strategies Used:

Rounded the 13 items to 50¢ each (13 x 50¢) = $6.50 then added the extra 2 items (2.50, 3.50) to obtain $12.50.

Exercise 15
Illustrative Subject Responses

Rounded to nearest dime unless exact number easily grouped with next number . . . started at top and added in order.

Rounded each number to nearest 10 or 5 cent amount (whichever easiest to add), kept cumulative total.

Grouped by dollar amounts in order, discounted ones along the way when felt he had an overestimate as subtotal.

Added dollar totals -- $7, then searched for places to group cents to dollar amounts. Verbalized problem of keeping track of used numbers.

Reviewed list of numbers, started cumulative total incorporating numbers which added "nicely" to subtotal (i.e. 2.29 + .11 + .08 gives about 2.50).

Looked at only first two digits (dollar digit and tenths or dime digit)- added these mentally keeping cumulative total.

Four addition problems were included in the interview battery. Exercises 1 and 9, each having relatively large addends, were similar in design.

Exercise 1	Exercise 9
87 419	73 655
92 765	86 421
90 045	91 943
81 974	97 509
+ 98 102	93 421
	+ 106 409

The most frequent strategy used in deriving the estimates to these problems involved averaging. It should be noted that there is little difference between the largest addend and the smallest, so the problems lend themselves to an averaging scheme. Another prevalent strategy used on these problems was rounding. Most subjects rounded to the ten-thousands place and proceeded to add from top to bottom. Over three-fourths of all subjects used one of these two strategies. Over 20 percent of those who used one of these schemes on Exercise 1 did not give an acceptable estimate. However, the frequency of unacceptable estimates on the first four exercises was greater than those on the remaining eleven. Thus, one might assume that factors other than the mathematical structure of the first four exercises confounded the early performance in the interviews. Such factors could include excitement, lack of adjustment to the interviewer and the surroundings, psychological unreadiness for this type of mental activity (there were no warm-up exercises), uncertainty of the demands of the task, and inability to accurately verbalize one's thoughts. During the interviews when subjects stated their estimates, the interviewer asked them to confirm their response. In repeating the process or estimate, the subjects often made corrections and/or adjustments to their initial response.

The other two addition problems included in the interview had several small addends and were presented in an applied, monetary context. Exercise 11 (a menu problem) had seven addends, each less than $3.00. Exercise 15 was a grocery ticket with twelve items ranging in size from 8¢ to $3.65. Very few subjects applied an averaging scheme to these problems. Instead, practically all of the subjects on Exercise 11 and about 90 percent of the subjects for Exercise 15 used a rounding strategy before adding. Apparently, the grocery ticket was conducive to a grouping strategy, inasmuch as about 40 percent of the subjects were observed grouping either two or three of the addends to obtain a convenient subtotal on their way to determining the estimate. About 40 percent used a grouping scheme with items on the menu. Table 12 confirms that subjects were highly successful in giving acceptable estimates to these two problems, with only one subject reporting an estimate out of the acceptable range on these two exercises.

The interview battery included two division problems, one in an applied context.

Exercise 3	Exercise 10
8 127 $\overline{)474\ 257}$	The 1979 Superbowl netted $21 319 908 to be equally divided among 26 NFL teams.

Rounding was used by over 60 percent of the subjects on each of these problems. About one-fourth of the subjects described some form of truncation while working the problems. In Exercise 3 they estimated the quotient of either 47 or 474 and 8 and then adjusted the place value of their partial quotient. Those using this strategy compensated giving an estimate over 50. Table 12 shows that 7 of the 15 subjects who used the truncation strategy on Exercise 3 were unsuccessful in giving an acceptable estimate. In fact, 38 percent of all subjects missed Exercise 3 regardless of the strategy used. Place value errors accounted for most of the unacceptable estimates. In Exercise 10 five of the subjects extracted several digits, formed an appropriate ratio, and reduced it to get an estimate of the quotient. All who attempted this technique gave acceptable estimates. In both division problems, compensation was readily used to adjust and/or refine estimates. Some form of compensation was used by about 40 percent of the subjects on Exercise 3 and 72 percent of the subjects on Exercise 10.

Seven of the problems in the interview battery involved at least one multiplication operation. The second exercise, 31 x 68 x 296, evoked a rounding strategy from all of the subjects. Over two-thirds of the subjects worked with 30, 70, and 300 to get their estimates. Although a majority of the subjects approached the problem in this manner, not all were successful in deriving acceptable estimates. Thirty-one percent of the subjects missed this problem - usually by a place value error. Over one-fourth of the subjects operated with the extracted front-end digits of rounded numbers, i.e., 3 x 7 x 3, and then added the appropriate number of zeros. This scheme was used by nearly half of the adults.

Exercise 4, (347 x 6)/43, evoked a great variety of identifiable strategies. It is predictable that a problem with multiple operations would elicit more strategies than a problem requiring only one operation. As in the previous problem, nearly all of the subjects rounded the three numbers in various ways and then performed the two operations. Three-fourths of the subjects multiplied and then divided. This approach was popular among all levels, which seemed a bit surprising. It was conjectured that arithmetic techniques (e.g. dividing a common factor) would be used by the older subjects to decrease the size of the numbers being computed, but Table 12 shows this approach was not popular.

Interview problems taken from the ACE Test included Exercises 5 and 6. Although Exercise 5 (1 7/8 x 1.19 x 4) was a difficult problem on the ACE Test, this was not the case during the interviews. Seventy-two percent of those interviewed missed this item on the ACE Test, but 96 percent answered it correctly during the interview. All respondents used some form of rounding strategy in deriving their estimate during the interview. The most popular, as well as the quickest strategy used was that of rounding to whole numbers. About 81 percent of the respondents rounded to whole numbers and/or decimals (2 x 1 x 4 or 2 x 1.2 x 4), and the remaining subjects rounded the numbers to fractions or a combination of fractions and decimals. Regardless of the strategy used in estimating, the subjects, as a group, were successful in giving an acceptable estimate.

Rounding was the most frequently observed strategy used in estimating the area of a rectangle 28 by 47

(Exercise 6). Over 80 percent rounded the numbers to 30 and 50, while the others sought more exact estimates by rounding only one of the numbers and then mentally computing. Over half of the respondents used some form of compensation after multiplying the two numbers together.

What are the proceeds from 8483 hotdogs at $0.60 each (Exercise 8)? Slightly more than half of the subjects (55 percent) used a rounded or truncated form of 8483 and multiplied by .60 or .6 to derive their estimates. Nearly one-third responded by converting the 60¢ to a fractional part of a dollar and multiplying by either 8000, 8400, or 8500. Subjects often checked their first estimate against ½ of 8483, a check which indicated if their estimate was reasonable.

Two of the problems involved the use of percentages. Exercise 7 called for 30% of 106,409 and Exercise 12 required 15% of $28.75. In Exercise 7, over one-half of the subjects converted the 30% to a decimal or fraction and then multiplied. A few (12 percent) decomposed 30% into 10% times 3. However, on Exercise 12 many more (47 percent) decomposed 15% into some form of 10% plus 5%.

Exercises 13 and 14 dealt with a common consumer problem - Which of the two cartons of soda has the most volume and which is the cheaper or better buy? One carton had 6-32 oz. bottles and the other had 8-16 oz. bottles. A large number of the subjects (39 percent) estimated the total number of ounces in each carton and compared the totals. About one-fourth computed the exact totals. Some subjects (14 percent) converted the volumes in the cartons to other units of measure - gallons, quarts or equivalent numbers of 16-ounce or 32-ounce bottles - for comparative purposes. About one-fourth of the subjects solved the problem by constructing a many-to-one correspondence between the smaller bottles in one carton and the larger bottles in the other.

Which carton of soda is the better buy? This was a challenging question that was answered correctly by 90 percent of the subjects. Not surprisingly, a majority (63 percent) of the subjects computed the unit price of each and compared. The unit pricing scheme led the subjects to conclude that one carton was priced at under 1¢ per ounce, whereas the other was a fraction over 1¢ per ounce. Twenty-two percent of the subjects effectively determined the better buy by computing a ratio of units in the two cartons and then compared the ratio (3:2) with the ratio of the prices. All who attempted this strategy were successful.

Among all of the items in the interview battery, this was the only one on which any of the subjects admitted they either guessed or didn't know the answer. Eleven percent of those interviewed either derived estimates that were too close for them to compare or had no strategy for using estimation to find a solution.

Summary of Strategies The front-end strategy characterized earlier (page 175) is one which was used by most of the subjects on several of the problems in the interview battery. The front-end strategy seemed to take on one of two forms. On some problems the subjects dropped one or more of the right-hand digits of the numbers. At this juncture either they operated on the extracted digits and adjusted the place value of their result or they replaced the digits that were dropped with zeros and then operated with the numbers in this form. This strategy was used by subjects on Exercises 1, 3, 9, 10, and 15. Those using this strategy were very successful in giving acceptable estimates for all of the exercises with the exception of Exercise 2. As reported earlier, the second exercise was difficult and was missed by nearly one-third of the subjects.

The other form of the front-end strategy was rounding. The subject rounded one or more of the numbers in the problem and then operated on this or an extracted portion of the rounded number. Subsequently, the result would be adjusted for place value. This strategy was used on a majority of the problems in the interview battery and is reported on each page of the description of exercise strategies as a rounding strategy.

The data from Table 12 were examined to see if differences in strategies used between age groups were apparent. The most notable instance of this occured in Exercise 11. For example, 8 of the 11 seventh-eighth graders rounded either to the nearest 5¢ or 10¢ amount, whereas the older subjects were less concerned with precision and rounded to larger amounts. This was the only obvious difference among age groups observed. For the most part, the most frequently used strategies for any problem were constant for each group.

Compensation was characterized on page 172 and described as either balancing adjustment of groups of numbers during the process of performing mathematical computations or an adjustment made to a preliminary result when it is recognized that this result is too large or too small.

The division of the proceeds from the 1979 Superbowl game (Exercise 10) by the 26 participating teams was a problem in which 72 percent of the subjects compensated by reducing their preliminary result. On Exercise 9 when estimating the total attendance at the games, most of the subjects who averaged the five addends to 100,000 and multiplied by 5, compensated the preliminary result by adjusting 500,000 downward to derive their estimate. Compensation used during the computation of the estimate was observed in problems having several addends (Exercises 1, 11, and 15) and in multiplication problems with three factors (Exercises 2 and 5).

Upon examination of the fifteen exercises in the interview battery, it becomes obvious that many subjects used compensation when estimating. It is noted that compensation was used by some subjects on nearly all of the described strategies. The amount of compensation was usually a function of the individual subject's understanding of number properties or desire for greater accuracy in estimation.

Discussion of Calculator Exercise

One of the characteristics of good estimators conjectured by these researchers was confidence in one's estimation skills. In order to test this conjecture several specially developed probes were built into both the straight computation and application interview problems. In addition, a specially designed calculator exercise was developed. This segment of the interview is outlined on page 32 of this report and is discussed below.

During the final portion of each interview, the subjects were reminded that up to this point they had been asked to provide estimates to a variety of problems without feedback regarding the accuracy of their estimates. Therefore, the last set of estimation problems included a provision for checking the accuracy of estimates with a calculator.

The HP-65 calculator used in this segment had been programmed to make systematic errors. The amount of error was increased as the subject worked successive groups of problems. Interviewers responded with specific probes as outlined on page 33. Once the subject verbalized that the calculator was in error, this portion of the interview was terminated. If the subject had not recognized the possibility of an error in the calculator by the seventh problem, then the interviewer asked leading questions until the subject recognized the calculator error.

It was hoped that this experience would provide insight into several very important and currently significant questions, including:

> Do good estimators have confidence in their estimates, even when confronted with conflicting evidence in the form of a calculator answer?
>
> Are good estimators sensitive to calculator errors?
>
> How large must the calculator error be before good estimators question the calculator answer?

The seven problems used in this exercise were designed with special considerations in mind. Each was constructed so that if common rounding strategies were employed, then the calculator would yield an answer above the upper boundary obtained by rounding numbers up. For example, in the first problem (436 + 972 + 79) if each addend were rounded up to the next hundred, the problem would be reformulated to 500 + 1000 + 100 or 1600. The calculator produced an answer of 1627. Likewise, in each of the remaining six problems, the calculator produced an answer greater than the expected upper bound.

On the first three problems the calculator produced an answer about 10% above the actual answer. If subjects proceeded to the fourth and fifth problems, a 25% error was added to the actual answer and finally, for the last two problems, the calculator produced a 50% error. Also, the calculator always produced an answer for which the right-most digit was correct. This was done to ensure that subjects would not be clued to detect the error based on exact computation starting with the right-hand digits.

Subjects greeted this segment of the interview with interest and enthusiasm. Generally, they were anxious to gain feedback on estimates. In all, 33 subjects in the student population and 12 adults were presented with this set of problems. Table 13 highlights how far each person worked before verbalizing the calculator error. Also provided are the percent of acceptable estimates given by these subjects. As reported in this table, 20 percent of the group recognized the unreasonableness of the result and voiced that the calculator-produced answer was wrong in the first exercise, while 36 percent of the subjects proceeded through the entire seven problems without verbalizing any concern about the accuracy of the calculator. As is recorded in the table, 5 of the 14 female subjects (36 percent) verbalized the calculator error before reaching the last problem while 24 of the 31 male subjects (77 percent) verbalized this error prior to reaching the final problem. What prompted this reluctance of females to challenge the calculator output is uncertain.

The following transcript of a ninth grader's reaction

Table 13

Number of Exercises Completed Prior to Verbalization of Calculator Error

Exercise (Acceptable Interval)		Frequency of Subjects Verbalizing Error at This Stage				Percentage of Responses Within Acceptable Interval	Percent of Subjects Verbalizing Error at this Stage	Percentage of Responses Within Acceptable Interval
10% error		7-8	9-10	11-12	Adult			
1. 436+972+79	M	1	3	1	4	93%	20%	93%
(1450 - 1600)	F	0	0	0	0			
2. 42 962 ÷ 73	M	0	0	0	0	64%	0%	64%
(550 - 650)	F	0	0	0	0			
3. 896 x 19	M	2	1	3	2	94%	22%	94%
(16 000 - 18 000)	F	0	0	2	0			
25% error								
4. 896+501+789	M	1	0	2	1	88%	11%	08%
(2000 - 2200)	F	0	0	0	1			
5. 28 x 47	M	0	1	1	0	95%	9%	95%
(1200 - 1500)	F	0	0	2	0			
50% error								
6. 22 x 39	M	0	1	0	0	88%	2%	88%
(800 - 900)	F	0	0	0	0			
7. 252 x 1.2 (or in a later	M	3	0	2	2	100%	36%	100%
discussion) (252 - 350)	F	1	2	4	2			
TOTAL		8	8	17	12			

M - Males
F - Females

to the calculator error on the first exercise is typical of those very confident subjects:

Transcript of Calculator Segment of a Ninth Grade Interview

Exercise: 436 + 972 + 79

Student Estimate: 1490   Calculator Response: 1627

Student: I messed up.

Interviewer: Do you want to write the calculator answer down, then we can try another one.

Student: That doesn't look right.

Interviewer: What do you mean?

Student: Well, it doesn't look like this is the right answer.

Interviewer: Should we try another one?

Student: Well, this is the wrong answer.

Interviewer: Why do you say that?

Student: Well, if this is 1000 + 436 + 79 it can't be 1600.

Interviewer: Do you think you pushed a wrong button?

Student: I guess so. (Uses calculator again to re-key the problem.)

Interviewer: So the answer is 1627?

Student: I know that's not right, you can tell by looking.

Interviewer: Are you sure?

Student: Yes.

Interviewer: What could be wrong?

Student: The calculator.

Interviewer: Have you ever used a calculator that was wrong?

Student: No, but say this is 80 + 970 = 1050 + 436. It can't be 1600.

Interviewer: So you say the calculator is wrong.

Student: Yes.

Interviewer: Maybe you did it wrong?

Student: No, I'm pretty sure I did it right?

A review of the estimates given by this group suggests that these subjects were making very good responses to each exercise. In fact, 88 percent of the estimates given were within an acceptable range previously established for these exercises.

Many subjects expressed puzzlement and hesitation when viewing many of the calculator responses but did not directly question them or verbalize their doubts. When this happened, the interviewer encouraged the subject to continue to the next problem. Sometimes when a person hesitated but then proceeded to the next problem, their doubts would be subdued by what they perceived to be a reasonable calculator answer on the next question. This was especially true after the first exercise.

Throughout this investigation, division was the most troublesome operation to deal with when estimating, especially when the numbers involved were large. This was evidenced by results in the second exercise in this calculator segment, which proved to be the most difficult. Many subjects made a place value error in their estimate. In setting up the problem, it was envisioned that subjects would see that the answer must be less than 600, while the calculator produced an answer of 638.52. However, this error was too subtle for most subjects. Perhaps because estimation with division is so difficult, subjects were willing to settle for any answer moderately close to their estimate. In only one instance did a subject hesitate and later verbalize that he doubted the calculator response to this problem. This twelfth grader stated, "It occurred to me that it (the calculator) smelled slightly here (division exercise). The result was larger than I expected (he had made an estimate of 580). Because it would seem to me that 73 adds a greater ratio over 70 than does 4300 over 4200 so that the answer should have been less than 600."

In other instances students waited until they had worked several problems before gaining confidence that the calculator was malfunctioning. "I suspected something was off on the third one, but I thought maybe I was messing up. After the second in a row, I knew I couldn't do that." (12th grader)

Subjects expressed doubt about the calculator response in several ways including puzzled looks, hesitancy to continue, desire to repunch the keystroking sequence, desire to use paper and pencil to calculate the answer and finally directly verbalizing that the calculator was wrong. Several comments are listed here which were common when doubts emerged:

> "I must have entered it wrong on the calculator; either that or I'm thinking wrong."
>
> "That (student's estimate) is kind of far off - 400 off. That wouldn't be a good estimate at all."
>
> "The battery could be bad, or I could have entered it wrong."
>
> "I don't understand. I don't understand what I'm doing wrong."
>
> "It doesn't look right, but if that's what the calculator says, then it's probably right. It still doesn't look right."
>
> "I'd like to enter that one again."
>
> "Can I work this one out on paper?"
>
> "I'm trying to figure out why it (calculator response) is 1627."
>
> "It shouldn't make a difference to work it out by hand, but I don't know."

> "The calculator is wrong - no, it couldn't be."
>
> "I thought my estimate would be really close, but it isn't."
>
> "I don't think the calculator seems reasonable."

Several complete transcripts of this segment of the interview are illustrated in Appendix 8 to provide more complete information on both how the interview was conducted and the types of reactions observed.

Any conclusions drawn from this segment would be tentative due to the small number of interviews as well as the nature of other variables which might have interfered with the students' verbalization of the calculator error. For example, the interviewer may have been viewed as an authority figure so that while students were willing to question the calculator, they were not comfortable in questioning the interviewer's techniques and authority. However, some general observations and comments do seem warranted and are offered.

1. Males were more likely than females to challenge the calculator result.
2. Even subjects making good estimates were reluctant to challenge the calculator result.
3. These results indicate that an aura of infallibility exists surrounding the calculator. Subjects who not only identified a clear discrepancy between their estimate and the calculator result and were able to explain why

the calculator result was too much, acquiesced and eventually accepted the calculator answer, claiming they must have made some mistake.

4. The unwillingness of these good estimators to reject unreasonable answers suggests that a challenging task lies ahead in preparing students to be alert to unreasonable answers.

Discussion of Concepts/Attitude Questions

In addition to asking subjects to formulate estimates and verbalize thinking processes, several additional questions were asked to clarify these good estimators' concept of estimation. The complete list of questions appears in Appendix 4, but responses to only some of them are reported here.

"Do you estimate?" was asked to determine whether subjects were aware of any use of estimation in their daily experiences. All but three of the forty-six students said yes. In a follow-up question, subjects were asked to describe situations they had encountered in which their estimation skills had been used. The most common application of estimation reported was in consumer settings. For example, subjects frequently cited estimating the total purchase price of items they wanted to buy. Several students also mentioned using estimation to check the reasonableness of answers obtained when using a calculator.

Although many varied and frequent uses of estimation were reported, less than half of the students said they used estimation in their mathematics classes. A ninth grade student, when asked if he used estimation in mathematics, said "Most of it (mathematics) is just getting the exact answer, it does not involve estimating." Those subjects who said they used estimation in mathematics class primarily reported using it to check their answers. However, a response given by an eleventh grade student characterized the feeling voiced by most students, "If I do (estimate), I don't think about it."

The use of estimation among adults was even greater. Although the specific use of estimation varied greatly, every adult identified frequent use of estimation in coping with real world applications such as: comparing prices while shopping; checking totals on a restaurant bill; determining gas mileage and estimating a checkbook balance. Some interesting uses of estimation were cited within different professions.

For example:

Engineer:

"I estimate the man-hours of work and the cost of those hours required to manufacture certain equipment parts. In this case we're working with very large numbers."

Physician:

"I estimate all the time. For example, after surgery I estimate fluid replacement. Like how many cc's a patient will need per hour per day. This is done all the time." "When prescribing antibiotics, you go on a weight basis and determine so many mg per kg and then you figure how much to give them each hour. So you break it down to rough estimates. The smaller the weight, the more critical the margin of error."

Mathematics Teacher:

"I estimate when making problems for a test and also in checking certain homework problems. It also helps me to see if I've made an error in computing. In working with logarithms, estimation helps me place the decimal point."

Bank Officer:

"In my work I estimate what mortgage payments might be or payments on different types of loans. I also estimate the amount of interest someone might pay in a given year or over the term of a loan."

In trying to determine the importance of estimation, the subjects were asked, "Is it (estimation) important?" Thirty-eight out of forty-six students said yes. A seventh grade student expressed the importance this way, "You can't always get an exact answer, yet you need a basis to make decisions." The importance was expressed by a twelfth grade student as, "You can't always have a calculator around, but you do need to do math quickly sometimes." The support for estimation among adults was even stronger, with each adult subject identifying estimation as an important skill.

When asked to define estimation, the subjects gave many different answers, although the definitions appeared to have some common themes.

The following terms were mentioned by many of the subjects in their descriptions of estimation:

1) reasonable close
2) fast
3) approximate (or rough)
4) computed mentally

It is both interesting and significant that these characteristics were included in the operational definition of computational estimation used in this investigation. It is felt that this agreement is both a reflection of the comprehensiveness of the proposed definition as well as its pragmatic nature. Such evidence helps validate several of the constructs associated with computational estimation that were reflected in its operational definition. A twelfth grade student showed considerable insight into the intricacies of computational estimation when he observed that estimation is dependent on the situation. More specifically he said estimation involves "trying to find an answer the quickest and most accurate way but it depends on the amount of time and importance of the objective."

In an attempt to learn where or how the students learned estimation, the question was asked, "Have you been taught how to estimate in school?" The predominant answer was that the students had been taught to round numbers, but that this skill was rarely used in conjunction with either the development or practice of estimation ability. Most students voiced uncertainty about where or how this skill had developed, frequently suggesting that they must have picked it up through the need for an efficient, reasonably accurate computational tool. Conversations with the adults provided similar information; they could not recall estimation being explicitly taught in school. Nevertheless, they developed many of these skills on their own. Thus, the interview data provide strong documentation that very little or no systematic instruction of estimation was experienced in schools by the subjects.

Respondents were also asked to describe techniques or strategies they use when estimating which might help another person to estimate.

Many of the hints were unique to a specific situation. Others provided some general heuristics and these are reported below.

- I. Prerequisites
  - A. Know basic bacts--"You have to know your basic facts."
  - B. Know properties of operations--"Know your mathematical rules, it will help to know what you can and can't do."
- II. Confidence--"Tell yourself not to be bothered by being off some."
- III. Reformulation
  - A. Front-end--"Deal with the big part of the number."
  - B. Rounding--"Round the numbers to the nearest multiple of ten."
  - C. Use easy numbers--"Sometimes I try to get nice numbers to work with."
  - D. Change the type of numbers to ones you can work with easiest--"In working with percents I change them to fractions. I can work with fractions easier."
- IV. Translations
  - A. Grouping--"Group by 'go-together' numbers."

B. Look for an easier way--"Break down the problem to easier problems."

V. Compensation--"Give or take from your answer--that's just your own judgement."

The first two categories deal with knowledge of prerequisite skills and having confidence in one's procedures. The remaining hints can be classified into the key processes proposed earlier and described more completely on pages 167 to 174.

Common Characteristics of Good Estimators

This research has collected much empirical data directly from students, adults and teachers as well as unobtrusive information from these same sources. The principal purpose of this research was to identify estimation strategies used by good estimators. This task produced many common as well as varied and unique strategies among the good estimators and has been reported.

A group of hypothesized characteristics were formulated in the early stages of this research effort and are identified in the following list (for a more complete description, see Table 1, page 24).

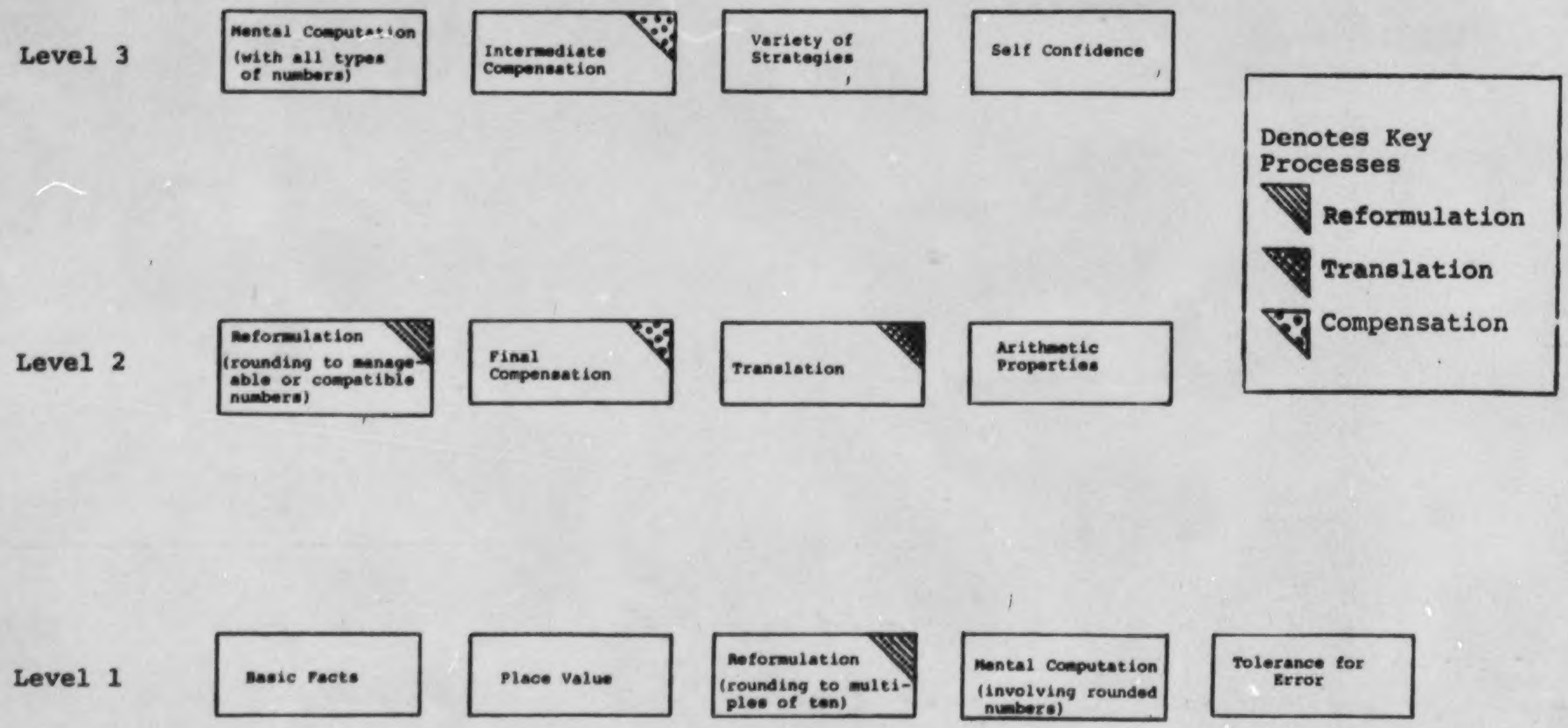
Quick with paper and pencil computations
Accurate with arithmetic computations
Unafraid to be wrong
Mathematical confidence

Demonstrated performance
Mathematical judgement
Reasonableness of answers
Divergent thinking strategies

A systematic validation of these characteristics was not the goal of this investigation. Rather, they were formulated to aid teachers in recommending interview candidates and in developing the interview protocols. Several, including reasonableness of answers, demonstrated performance and mathematical judgement were clearly outside the scope of validation within a limited interview session. These characteristics, however, did provide a basis for helping to organize the interview data. In addition, following the interviews, the investigators asked themselves the following questions: Are there specific qualities or traits associated with these subjects? Do these good estimators call upon specific skills and abilities which contribute to their estimation success? If so, can these characteristics be identified? The data suggest an affirmative answer to each of these questions. This information, together with general discussions among interviewers and informal conversations with the students' mathematics teachers, provided additional insight into the search for common characteristics. The most frequent characteristics exhibited are presented in Table 14. This model provides

Table 14

Identifiable Characteristics Associated With Good Estimators



Level 3	Mental Computation (with all types of numbers)	Intermediate Compensation ⁙	Variety of Strategies	Self Confidence	
Level 2	Reformulation (rounding to manageable or compatible numbers) ▧	Final Compensation ⁙	Translation ▩	Arithmetic Properties	
Level 1	Basic Facts	Place Value	Reformulation (rounding to multiples of ten) ▧	Mental Computation (involving rounded numbers)	Tolerance for Error

Denotes Key Processes

▧ Reformulation

▩ Translation

⁙ Compensation

a skematic overview of essential characteristics (factors or constructs) that have been associated with people possessing exceptional computational estimation skills. The coded cells in Table 14 illustrate characteristics associated with one of the three key processes described earlier in this report.

The three levels present a hierarchical arrangement reflecting the existence of characteristics as follows:

Level 1 -- each characteristic present among every subject interviewed.

Level 2 -- each characteristic present among a majority of subjects interviewed at every level.

Level 3 -- each characteristic present among 20 to 50 percent of the subjects interviewed at every level.

Here is a brief description of each identified characteristic.

Basic Facts - Subjects possessed a quick and accurate recall of basic facts for all operations.

Place Value - Subjects possessed a good sense of how place value is affected by different operations of arithmetic. Using this knowledge allowed the subject more flexibility in choice of an estimation strategy, as well as more assurance of accuracy.

Reformulation - Changing the numerical data within a problem to a mentally manageable form was common. This form in some cases involved rounding numbers to multiples of ten. Less common, but very effective, was the use of rounding to convenient multiples of existing numbers in the problem. Numbers were also rounded to approximately equivalent forms (e.g. decimals to fractions).

Mental Computation - Common to the majority of subjects interviewed was the quick and efficient use of mental computation to produce accurate numerical information with which to formulate estimates. All subjects exhibited well developed skill with multiples of ten or a limited number of digits, while many others were fluent in mentally computing with larger numbers, more digits and even different types of numbers (e.g. fractions). On some problems, subjects resorted to mental computation rather than utilization of an estimation technique. For these problems and these subjects, it was more efficient for the person to mentally compute rather than estimate. Whatever the level of mental computation ability, it aided the estimator in managing the numerical data presented.

Tolerance for Error - Knowledge of what estimation is was found to permeate the thinking of good estimators. This understanding of the concept of an estimate enabled them to be comfortable with some error. They frequently noted the importance of an efficient, reasonably accurate computational tool and felt that their ability to estimate filled this need. In other words, they saw estimation as an important tool when dealing with numbers and didn't see themselves as being "wrong" when using estimates. One seventh grade student, in offering hints to improve estimation ability, said it was important to " . . . tell yourself not to be bothered by being off some."

Compensation - The ability and insistence to adjust an initial estimate to reflect numerical variation which came about as a result of translation and/or reformulation of the problem was found among many subjects. While the degree of use and form of compensation varied among subjects, the majority voiced the importance of this process, identified it as an essential component to any estimation strategy and recommended it to those less proficient in estimating.

Translation - Subjects often approached a problem by translating it to a more manageable form. Unlike reformulation, where only numbers were changed, a translation involved both changing numbers as well as changing the mathematical structure of the equation or problem. For example, an addition problem involving several addends of similar value might be translated to a multiplication problem where an approximate average of the addends was multiplied by the number of addends. This process was most noticeable when the problem involved more than one operation, a group of large numbers or when the estimators capitalized on their own particular computational strengths.

Arithmetic Properties - Many subjects possessed a knowledge and use of number properties including distributive, associative and commutative properties. In addition, they exhibited appropriate choices which reflected a knowledge of order of operations. Use of these properties was rapid and concise. This command of a variety of computational tools allowed the subject valuable flexibility in choosing the estimation strategy to be used.

Variety of Strategies - Some subjects demonstrated that they possessed a variety of strategies and techniques to attack any given problem. Several indicated that before beginning to formulate an estimate, they quickly sorted through several strategies that come to them in a search for a quick and accurate method. This search was internal and immediate. The particular estimation strategy used for a problem depended on the numbers and operations involved. In several instances, subjects stated that they "switched" strategies when a particular thought process was unproductive. One subject declared, "If my first way causes the numbers to get out of hand, I start over and think of a different way."

Self Confidence - In an effort to document the level of confidence in their own estimation ability, a variety of techniques were employed in the interview. One of these, the calculator segment, is discussed elsewhere in this report and documents that levels of confidence vary among these able estimators. For example, subjects were confident about their estimation ability, although this confidence often weakened when confronted with conflicting evidence. Subjects were asked on several interview problems to indicate how confident they were they had made a good estimate using a semantic differential scale. The confidence exhibited here tended to depend on the particular problem; however, responses were typically toward the "certain" pole of the scale.

> Some of the subjects retained a strong self confidence in both the strategies they were using and the estimates they gave throughout the entire interview. These subjects were typically the first to challange the calculator output in that portion of the interview. This limited group of subjects were confident in their own estimation ability and this confidence influenced their consistency, quickness and choice of strategy. They seemed to understand clearly the concept of an estimate and were comfortable in producing estimates which were frequently less accurate than other subjects' responses.

These characteristics represent three distinct dimensions:

1. number skills
2. cognitive processes
3. affective attributes.

The data from this research collectively support the model shown in Table 14 and confirm that good computational estimators do indeed exhibit these specific characteristics. The model also confirms that not every person interviewed reflected every component. In fact, some components from Level 2 and 3 were conspiciously absent in several interviews. Why? Perhaps the interviews lacked sufficient depth to document the presence of these constructs or perhaps the analysis of the interview data was not sensitive enough to detect their existence. Documenting these constructs is also confounded by the fact that they are not dichotomous attributes but each is distributed along a continuum.

Much additional research is needed to validate or reject the framework proposed. The processes necessary to verify the proposed constructs are both varied and complex. The procedures used in this project (synthesis of related research, individual interviews and discussion with classroom teachers, researchers and other mathematics educators) were productive. It is hoped that the procedures as spelled out in this report will allow for different replications as deemed appropriate by other researchers. In the process it is anticipated that these procedures can be refined, revised and improved upon to either verify or repudiate this model. In this spirit, the closing section offers suggestions for further research.

Questions For Further Study

This research has raised many significant and researchable questions. In the hope of both encouraging and promoting more research studies in this important area, the following suggestions are offered.

1. A factor analytic study of identifiable characteristics associated with good estimators is needed. Such research should not only check on the existence of the proposed constructs in the model developed in this research project but examine the existence of distinct factors as well as the degree and/or weight that should be associated with each of them.
2. A systematic plan of research to study the relationship between the strategies used and variables associated with the computational estimation problems is needed. This project revealed that various strategies were used on different problems during the interview. Although this study was not structured to systematically examine the affect of different variables (e.g. size and type of numbers involved, the operations involved and the format for problem presentation)

on estimation strategies actually used, such research needs to be done.

3. A study to examine the large sex differences in computational estimation and seek explanations for these performances is needed. A disproportionate number of males were identified as good estimators by the ACE Test. This performance difference was also supported by the interview data in which males consistently performed better than females. Perhaps research directed toward this issue would produce some plausiable explanations for the dramatic sex differences reported here.

4. Research to learn more about sensitivity to unreasonable answers and techniques used to identify out-of-the-ballpark answers is needed. This is a complex phenomona to research and this project did not include a systematic effort to survey it. However, instances occurred in which subjects quickly changed their estimate because they perceived it to be unreasonable. This occurred predominately with

division in which estimates were made which contained a place value error. For example, on Exercise 3 of the interview ($8{,}127\overline{)474{,}257}$) estimates included 60, 600, 6000 and even 60,000. While some subjects quickly rejected their initial estimate of 60,000 as "not sounding right" others were content with what they perceived as a good estimate to a difficult problem.

5. A thorough study of the relationship between subjects' self confidence toward estimation and their performance on computational estimation is needed. Such research must develop sensitive measures of self confidence.

6. A systematic plan for teaching estimation to students in grades 7-12 including the most prevelant and effective techniques and strategies identified by this investigation should be developed and tested. Can students with only minimum estimation ability be taught to use strategies identified among good estimators? What subskills identified in the framework proposed in Table 14 can effectively be taught and used by students?

REFERENCES

Bell, M. S. What does 'everyman' really need from school mathematics? Mathematics Teacher, March 1974, 67, 196-202.

Bestgen, B. J., Reys, R. E., Rybolt, J. F., Wyatt, J. W. Effectiveness of systematic instruction on attitudes and computational estimation skills of preservice elementary teachers. Journal for Research in Mathematics Education, March 1980, 11, 124-136.

Brown, B. I. A study in mental arithmetic: proficiency and thought processes of pupils solving subtraction examples. (Doctoral dissertation, University of Pittsburg, 1957.) Dissertation Abstracts International, 1957, 17, 2219A.

Buckley, P. B. Two processes for mental arithmetic (Doctoral dissertation, University of Wisconsin-Madison, 1974.) Dissertation Abstracts International, 1974, 36, 0466B.

Buchanan, A. O. Estimation as an essential mathematical skill (Professional Paper 39). Los Angeles: Southwest Regional Laboratory for Educational Research and Development, August 1978.

Carpenter, T. P., Coburn, T. G., Reys, R. E., and Wilson, J. W. Results from the First Mathematics Assessment of the National Assessment of Educational Progress. Reston, VA: National Council of Teachers of Mathematics, 1978.

Carpenter, T. P., Coburn, T. G., Reys, R. E., and Wilson, J. W. Notes from national assessment: estimation. Arithmetic Teacher, April 1976, 23, 297-302.

Carpenter, T. P., Corbitt, M. K., Kepner, H., Lindquist, M. M., and Reys, R. E. Results and implications of the second NAEP mathematics assessment: elementary school. Arithmetic Teacher, April 1980, 27, 10-12+

Corle, C. G. Thought processes in grade six problems. Arithmetic Teacher, October 1958, 5, 193-203

Dickey, J. W. The value of estimating answers to arithmetic problems and examples. Elementary School Journal, September 1934, 35, 24-31.

Faulk, C. J. How well do pupils estimate answers? Arithmetic Teacher, December 1962, 9, 436-440.

Flournoy, M. F. Developing ability in mental arithmetic. Arithmetic Teacher, October 1959, 6, 133-139. (a)

Flournoy, M. F. Providing mental arithmetic experiences. Arithmetic Teacher, April 1959, 6, 133-139. (b)

Freeman, D., Kuhs, T., Belli, G., Floden, B., Khappen, L., Porter, A., Schmidt, B., and Schwille, J. The fourth grade mathemtaics curriculum as inferred from textbooks and tests. Paper presented at 1980 Annual Meeting, American Educational Research Association: Boston, Massachusetts.

Good, C. V. Dictionary of Education (3rd edition). New York: McGraw-Hill, 1973

Hall, J. V. Business uses of mental arithmetic in Ellensburg, Washington. Unpublished doctoral dissertation, University of Northern Colorado, 1951.

Josephina, Sister. Mental arithmetic in today's classroom. Arithmetic Teacher, April 1960, 7, 199-200.

Lawson, T. J. A study of the calculator's and altered calculator's effect upon student perception and utilization of an estimation algorithm. (Doctoral dissertation, State University of New York at Buffalo, 1977). Dissertation Abstracts International, 1978, 39, 647.

National Council of Supervisors of Mathematics. Position paper on basic mathematical skills, January 1977.

National Council of Teachers of Mathematics, An Agenda for Action: Recommendations for School Mathematics of the 1980's. Reston, VA: The Council, 1980.

Nelson, N. Z. The effect of the teaching of estimation on arithmetic achievement in the 4th and 6th grades (Doctoral dissertation, University of Pittsburgh, 1966). Dissertation Abstracts International, 1967, 27, 4127A.

NIE Conference on Basic Mathematical Skills and Learning (2 vols.). Washington, D.C.: National Institute of Education, 1975.

Olander, H. T. and Brown, B. I. A research in mental arithmetic involving subtraction. Journal of Educational Research, November 1959, 53, 97-102.

Olshen, J. S. The use of performance models in establishing norms on a mental arithmetic test (Doctoral dissertation, Stanford University, 1975). Dissertation Abstracts International, 1975, 36, 6033A.

Paull, D. R. The ability to estimate in mathematics (Doctoral disseration, Columbia University, 1971). Dissertation Abstracts International, 1971, 32, 3567A.

Sachar, J. An instrument for evaluating mental arithmetic skills. Journal for Research in Mathematics Education, May 1978, 9, 233-237.

Skvarcius, R. The place of estimation in the mathematics curriculum of the junior high school. Cape Ann Conference on Junior High School Mathematics. Boston: Physical Science Group, 1973.

Trafton, P. R. Estimation and mental arithmetic: important compenents of computation. In M. N. Suydam and R. E. Reys (Eds.), Developing Computational Skills, 1978 Yearbook of the National Council of Teachers of Mathematics. Reston, VA: The Council, 1978.

Urbatsch, T. D. The effects of instruction in product estimation on computation, estimation and problem solving skills of fourth graders. Paper presented at the meeting of National Council of Teachers of Mathematics, Des Moines, IA, April,1979.

Wilson, J. W., Cahen, L. S. & Begle, E. G. (Eds.). *NLSMA Reports* (Nos. 1A, 2A, 4, 5), Palo Alto, CA: School Mathematics Study Group, 1968.

OTHER REFERENCES

Boulware, C. E. The emerging concept of mental arithmetic. Unpublished doctoral dissertation, Columbia University, 1950.

Bright, G. W. Estimation as part of learning to measure. In D. Nelson and R. E. Reys (eds.), *Measurement in School Mathematics*, 1976 Yearbook of the National Council of Teachers of Mathematics. Reston, VA: The Council, 1976.

Carlton, R. A. Basic skills in the changing world. Paper presented at Basic Skills for Productivity and Participation Conference. National Institute of Education, University of Guelph, May 1980.

Carpenter, T. P., Corbitt, M. K., Kepner, H., Lindquist, M. M., and Reys, R. E. *Results fron the Second Mathematics Assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics, in press.

Corle, C. G. Estimates of quantity by elementary teachers and college juniors. *Arithmetic Teacher*, October 1963, *10*, 347-352.

Damgaard, G. T. A study to determine the significance of a mental arithmetic program for grade six. Unpublished dissertation, University of Northern Colorado, 1958.

Driscoll, M. Estimation and mental arithmetic. *Research Within Reach*, R & D Interpretation Service CEMREL Inc., St. Louis, MO, December 1979.

Eickman, M. A. The effect of consumer context and practice on the acquisition of numerical estimation skills at the junior high level (Doctoral dissertation, University of Minnesota, 1979). *Dissertation Abstracts International*, 1979, *40*, 4937A.

Flournoy, M. F. The effectiveness of instruction in mental arithmetic. *Elementary School Journal*, November 1954, *55*, 148-153.

Ginsburg, H. Young children's informal knowledge of mathematics. *Journal of Children's Mathematical Behavior*, Summer 1975, *1*, 63-156.

Grumbling, B. L. An experimental study of the effectiveness of instruction in mental computation in grade four (Doctoral dissertation, University of Northern Colorado, 1970). *Dissertation Abstracts International*, 1970, *31*, 3775A.

Hall, D. E. The ability of intermediate grade children to deal with aspects of quantitative judgement. Unpublished doctoral dissertation, Boston University, 1965.

Hall, J. V. Mental arithmetic: misunderstood terms and meanings. *Elementary School Journal*, February 1954, *54*, 349-353.

Hall, J. V. Solving verbal arithmetic problems without pencil and paper. *Elementary School Journal*, December 1947, *48*, 212-217.

Hauk, C. M. Grade Six Students' Methods of Estimating Answers to Computational Exercises. Unpublished master thesis, University of Alberta, 1978.

Johnson, D. C. Teaching estimation and reasonableness of results. *Arithmetic Teacher*, September 1979, *27*, 34-35.

Koenker, R. H. Mental arithmetic. *Grade Teacher*, September 1963, *81*, 128-131.

Dramer, K. Adding and subtracting without pencil and paper. In K. Kramer (Ed.), *Problems in the Teaching of Elementary School Mathematics*. Boston: Allyn and Bacon, 1970.

Kropp, R. P. An evaluation of two methods of test interpretation and the related analysis of oral problem-solving processes. Unpublished doctoral dissertation, University of Illinois, 1953.

Moser, H. E. Developing the ability to compute mentally. *Journal of Education*, December 1953, *136*, 79-82.

National Assessment of Educational Progress. *The Second Assessment of Mathematics, 1977-78: Released Exercise Set*. Denver: Education Commission of the States, 1979.

Newcombe, R. S. Teaching pupils how to solve problems in arithmetic. *Elementary School Journal*, November 1922, *23*, 183-189.

O'Daffer, P. A case and techniques for estimation: estimation experiences in elementary school mathematics - essential, not extra! *Arithmetic Teacher*, February 1979, *26*, 46-53.

Page, D. Do something about estimation. Updating Mathematics. University of Illinois Arithmetic Project, Vol. II, No. 8. Urbana, Illinois: University of Illinois, 1960.

Payne, J. F. An experimental study on the effectiveness of instruction in mental computation in grade four (Doctoral dissertation, University of Northern Colorado, 1966). Dissertation Abstracts International, 1966, 27, 0608A.

Payne, J. and Rathmell, E. Number and numeration. In Joseph N. Payne (Ed.), Mathematics Learning in Early Childhood, 37th Yearbook of the National Council of Teachers of Mathematics. Reston, VA: The Council, 1974.

Petty, O. Lay that pencil down. Grade Teacher, May 1957, 74, p. 57.

Rathmell, E. C. Using thinking strategies to teach basic facts. In M. Suydam and R. Reys (Eds.), Developing Computational Skills, 1978 Yearbook of the National Council of Teachers of Mathematics. Reston, VA: The Council, 1978.

Sauble, I. Development fo ability to estimate and to compute mentally. Arithmetic Teacher, April 1955, 2, 33-39.

Schall, W. E. A comparison of mental arithmetic modes of presentation in elementary school mathematics (Doctoral dissertation, The Pennsylvania State University, 1969). Dissertation Abstracts International, 1969, 31, 0684A.

Schall, W. E. Comparing mental arithmetic modes of presentation in elementary school mathematics. School Science and Mathematics, May 1973, 73, 359-367.

Sutherlin, W. N. The pocket calculator: its effect on the acquisition of decimal estimation skills at intermediate grade levels (Doctoral dissertation, University of Oregon, 1976). Dissertation Abstracts International, 1977, 37, 5663A.

Trimble, H. C. Teaching about "about". Arithmetic Teacher, February 1973, 20, 129-133.

Wandt, E. and Brown, G. W. Non-occupational uses of mathematics. Arithmetic Teacher, October 1957, 4, 147-154.

Washburne, C. W. and Osborne, R. Solving arithmetic problems. Elementary School Journal, December 1926, 27, 219-226+.

West, R. L. and Shuster, C. N. The teaching of approximate computation in elementary grades. Education, April 1941, 61, 492-497.

Wilson, E. Improving the ability to read arithmetic problems. Elementary School Journal, January 1922, 22, 380-386.

Winkelman, J. H. The repetition effect in mental arithmetic: a speed-accuracy study (Doctoral dissertation, University of Oregon, 1974). Dissertation Abstracts International, 1974, 35, 6154B.

Wise, C. T. A survey of arithmetic problems arising in various occupations. Elementary School Journal, October 1919, 20, p. 118.

Zepp, R. Algorithms and mental computation. In M. Suydam and A. Osborne (Eds.), Algorithmic Learning. The ERIC Science, Mathematics and Environmental Education Clearinghouse ED 113 152.

APPENDIX 1

ACE Test Answer Sheet

Front side

Name ____________
Grade ____ Sex ____
Class ____________

A. ____________
B. ____________

---

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________

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6. ____________
7. ____________
8. ____________
9. ____________
10. ____________

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11. ____________
12. ____________
13. ____________
14. ____________
15. ____________

---

16. ____________
17. ____________
18. ____________
19. ____________
20. ____________

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21. ____________
22. ____________
23. ____________
24. ____________
25. ____________

---

26. ____________
27. ____________
28. ____________

Back side

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________

---

6. ____________
7. ____________
8. ____________
9. ____________
10. ____________

---

11. ____________
12. ____________
13. ____________
14. ____________
15. ____________

---

16. ____________
17. ____________
18. ____________
19. ____________
20. ____________

---

21. ____________
22. ____________
23. ____________
24. ____________
25. ____________

---

26. ____________
27. ____________
28. yes no not sure

APPENDIX 2

Directions for Group Administration of ACE Test

Directions for Group Administration of ACE Test

Ask students to:

1. Clear desk completely except for a ball point or felt tipped pen.

2. Position themselves so they can clearly see the screen.

-----Distribute answer sheets-----

3. Fill out information at top of side one of answer sheet.

"I'd like you to estimate the answer to some computation problems. Each problem will be presented on a slide which will be projected on the screen. You'll have a limited amount of time to determine about what each answer is so it is important that you estimate rather than try to determine an exact answer."

"The first 28 exercises of this estimation test are straight computation problems. You'll see an addition, subtraction, multiplication, or division exercise projected. Write only your estimate on this answer sheet. You are to make no other marks or recordings. Do you understand?"

"You will be given anywhere from 10(12) to 15(17) seconds to make your estimate with more difficult exercises receiving a longer allotment of time. You'll be given 2-3 seconds to record your estimate after each question. In order to give you a feel for about how much time you'll be allowed, the first two slides are only examples. Record your estimate on lines A and B for these exercises. Are you ready?"

-----Administer the sample exercises-----

-----Check during the first slide to be sure everyone can see.

"Are there any questions? Remember, you are to estimate. If you can mentally compute the answer, fine--otherwise estimate as close as you can."

-----Start slides and time each carefully-----

-----At end of the 28th exercise ask subjects to turn their papers over.-----

"The next set of slides are different. Again, however, I want you to estimate each answer. You will be presented a problem situation. For example, in the first slide, you'll see some apples and a price per apple and will be asked to estimate the total for purchasing all the apples. These slides will also be timed."

-----Start slides, time each carefully-----

-----At end of slides:

"Please pass your answer sheets to the front."

APPENDIX 3

Characterization of Levels of Estimation Skill

Characterization of Levels of Estimation Skill

1. No evidence of estimation skills--consistently uses computation when confronted with an estimation task; does not give any evidence of varying estimation strategies.

2. Minimal or primitive estimation skills--occasionally uses estimation strategies; will estimate when "has to"; tends to fall back on exact computation; reveals very few estimation strategies; simple strategies only; limited confidence in estimates.

3. Functional estimation skills--uses estimation strategies; limited flexibility; not necessarily very accurate; some degree of confidence in estimates.

4. Good estimation skills--evidence of a variety of estimation strategies; usually accurate; confidence in accuracy of the estimates and confidence in the processes used.

5. Very proficient estimation skills--evidence of a variety of estimation strategies; reasonably accurate estimates; very confident in the accuracy; very confident in the process.

APPENDIX 4

Interview Packet

- - - - - -

"My name is _____. Do you remember when I came into your classroom and gave the class the estimation test? In looking over the estimates that each of you gave, I noticed that you did a good job and I'd like to ask you a few more estimation questions if I might."

ESTIMATION

- - - - - -

* Do you remember the estimation test you took in class? Do you ever do this kind of thing in or out of school? In mathematics class? (At work? As a consumer?)

* Do you think estimation is part of mathematics?

* Do you think estimation is an important skill?

"I have been interviewing students like yourself in the last few weeks in an attempt to identify what strategies or methods students use when they estimate the answer to certain questions. I'm going to show you a few problem situations and I'd like you to estimate the answer to each. As you estimate, I want you to tell me what you are thinking. This will help me understand how you arrive at your estimate. You may not think some of the things are important but they may help me understand what you are thinking so please think out loud. Do you understand?"

$$\begin{array}{r} 87\ 419 \\ 92\ 765 \\ 90\ 045 \\ 81\ 974 \\ +\ 98\ 102 \\ \hline \end{array}$$

_ _ _ _ _ _

P_4* -- "Do you think the actual answer is above or below your estimate?"

---

* Notation refers to standard probes identified on page of Interview Packet.

31 x 68 x 296

- - - - - -

P_4: "Do you think the actual answer is above or below your estimate?"

$$8\,127\ \overline{)\,474\,257}$$

_ _ _ _ _ _

$$\frac{347 \times 6}{43}$$

- - - - - -

P_2: "Is there another way you could do that?"

$1\frac{7}{8} \times 1.19 \times 4$

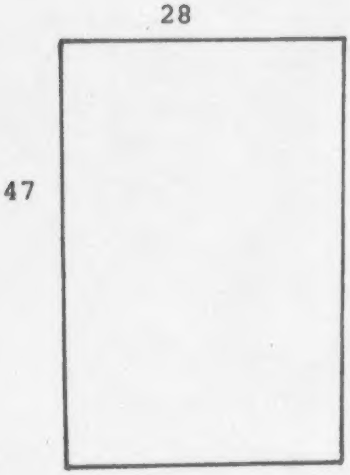
_ _ _ _ _ _

Unsure Certain

(Flip when estimate made and explanation complete.)

P_3: "How confident are you that you've made a good estimate." (Explain scale.)

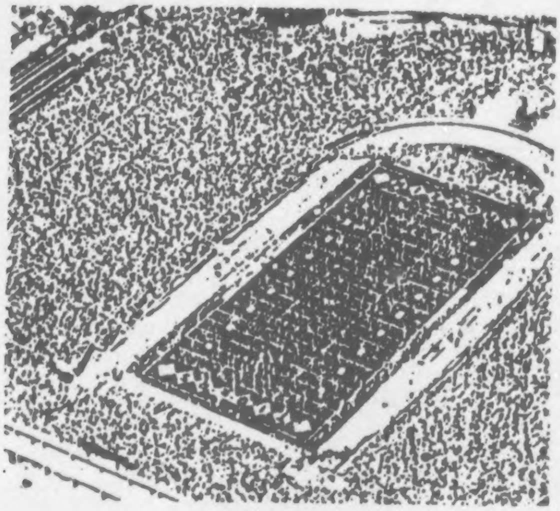
About how much area does
this rectangle have?



- - - - - -

P_1: "A student I interviewed last week estimated this area to be 2000--is that a good estimate?"

If Yes: Why?
If No : What would be the largest estimate that you would accept as a good estimate? Why?



If 30% of the fans at the 1979 Superbowl bought one soda, about how may sodas were sold at that game?

Year	Attendance
1974	73 655
1975	86 421
1976	91 943
1977	96 509
1978	93 421
1979	106 409

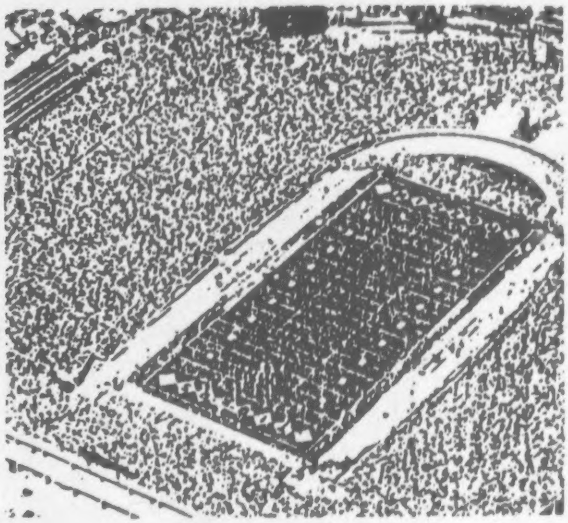
- - - - - -

P_4: "Will your estimate be more or less than the actual answer?"

At the 1979 Superbowl 8 483 hot dogs were sold for $.60 each. About how much resulted from selling the hot dogs?

- - - - - -

P_2: "Can you think of another way to do that problem?"



Here are 3 estimates for the total attendance for the past 6 Superbowl games:

1 000 000
600 000
550 000

Year	Attendance
1974	73 655
1975	86 421
1976	91 943
1977	96 509
1978	93 421
1979	106 409

- - - - - -

Ask student: "Which is the best estimate?"

The 1979 Superbowl netted $21 319 908 to be equally divided among the 26 NFL teams. About how much does each team receive?

_ _ _ _ _ _

Three people have dinner. They order:

Bacon n Cheese Steakburger Platter
Super Steakburger Platter
Chili-Mac
2 small Coca-Cola's
1 hot chocolate
1 hot pie

About how much money will be needed to pay the bill?

- - - - - -

(Give the subject Steak n Shake menu.)

P_1: "Another student estimated the bill to be $11. Is that a better estimate than yours?"

PLATTER SPECIALS

All our Sandwich Platters are served with our own golden brown French Fries and your choice of Lettuce and Tomato Salad or Baked Beans

BACON n CHEESE STEAKBURGER PLATTER Featuring a big one-third pound *Steakburger, covered with melted real cheese and lots of crispy real bacon	$2.89
STEAKBURGER PLATTER Features the original Steakburger sandwich FAMOUS SINCE 1934 — served on our own delicious toasted bun with your choice of sandwich dressings	$1.99
WITH MELTED CHEESE	$2.14
SUPERSTEAKBURGER PLATTER Double Delicious with Two Steakburger patties served on our own delicious toasted bun with your choice of sandwich dressings	$2.64
WITH MELTED CHEESE	$2.79
BAKED SUGAR CURED HAM (Hot or Cold) A generous portion of our own famous oven baked ham served on a toasted bun	$2.89
WITH MELTED CHEESE	$3.04

LO-CAL PLATTER Two Steakburger patties served with sliced tomato, lettuce and cottage cheese	$2.09

DELUXE SANDWICHES

All our Steakburgurs are made with Government inspected 100% pure beef, including such fine steak cuts as T-bone, strip steaks and sirloin.

STEAKBURGER — FAMOUS SINCE 1934	89¢
WITH MELTED CHEESE	$1.04
SUPERSTEAKBURGER — Two Steakburger patties	$1.54
WITH MELTED CHEESE	$1.69
BACON n CHEESE STEAKBURGER — 100% delicious	$1.79
BAKED SUGAR CURED HAM (Hot or Cold)	$1.79
WITH MELTED CHEESE	$1.94
TOASTED CHEESE Two slices of American Cheese on toasted bread, grilled in butter	78¢

Sandwich Dressings: Onion, Pickle, Relish, Mustard, Catsup, 1000 Island, Mayonnaise

LETTUCE AND TOMATO ON ANY SANDWICH	Add 16¢

CHILI SPECIALTIES

All our Chili Specialties start with 100%, ground, top round steak. Our kidney beans are plump, red, simmered for hours. Our chili sauce is a special blend of tomato with zesty spices.

CHILI — Our Own Genuine You will like it	$1.25
CHILI-MAC Liberal order of Italian Spaghetti and Chili Meat	$1.37
CHILI-THREE WAYS Italian Spaghetti, Chili Beans and Chili Meat	$1.75
extra meat available on the above items	62¢

*Weight before cooking

SPECIALTIES

FRENCH FRIES Golden brown sizzling hot, a liberal order	60¢
BAKED BEANS Our Own Special Sauce, Individual Pot	72¢
COTTAGE CHEESE Served with sliced tomatoes and lettuce	72¢
LETTUCE AND TOMATO SALAD Choice of dressing	72¢
CHEF'S SALAD Julienne of ham, chicken and cheese garnished with tomato wedges and egg slices. Onion available upon request. Choice of dressing	$2.21

Salad Dressings: French, Creamy Italian, 1000 Island, Bleu Cheese, Mayonnaise

DESSERTS and DRINKS

STRAWBERRY SUNDAE Delicious, Topped with real Strawberries, whipped topping and a maraschino cherry	92¢
HOT FUDGE NUT SUNDAE With plenty of rich chocolate fudge, nuts, whipped topping and a maraschino cherry	92¢
BROWNIE FUDGE SUNDAE Our Brownie topped with ice cream, rich chocolate fudge, whipped topping and a maraschino cherry	92¢
HOT-PIE Dutch Apple, Dutch Cherry and Southern Pecan	76¢
ALA MODE	94¢
CHEESE-CAKE A Favorite, Our Own Creamy, Delicious	84¢
CHEESE-CAKE WITH STRAWBERRY TOPPING Even Better	95¢
OUR FAMOUS VANILLA ICE CREAM	50¢
BROWNIE Our own butter-rich fudge brownie	50¢
DANISH Apple or Cinnamon. Served warm with butter	49¢

TRU-FLAVOR MILK SHAKES Made the Old Fashioned Way. Chocolate, Vanilla, Strawberry	Reg. 90¢
FREEZES Orange or Lemon. A Year Round Flavor Treat	75¢ and 90¢
FLOATS Coca-Cola, Root Beer, Orange or Lemon	75¢

FRUIT DRINKS Orange or Lemon	45¢ and 55¢
Coca-Cola	45¢ and 55¢
Sprite	45¢ and 55¢
FANTA ROOT BEER	45¢ and 55¢
TaB	45¢ and 55¢

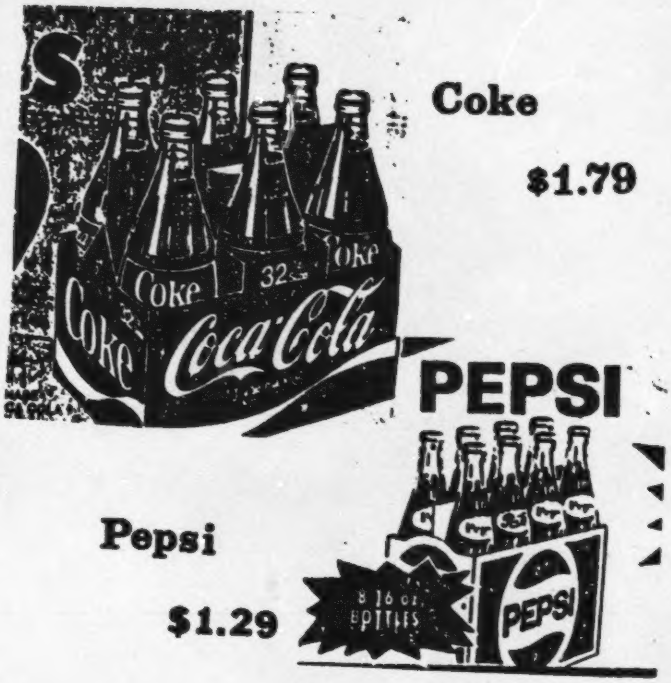
COFFEE A Steak n Shake Specialty	35¢
HOT TEA Individual Pot	35¢
ICED TEA	45¢
MILK	45¢ and 60¢
HOT CHOCOLATE	35¢

The Thompson's dinner bill totaled $28.75. Mr. Thompson wants to leave a tip of about 15%. About how much should he leave for the tip?

- - - - - -

P_2: "Is there another way you could do that?"

Which carton has
more soda?


Coke
32
Coca-Cola
PEPSI
8 16 oz. BOTTLES
PEPSI

Coke

$1.79

Pepsi

$1.29

______

Which soda is the cheapest?

Coke
Coca-Cola
32

Coke

$1.79

PEPSI
6 16 oz BOTTLES
PEPSI

Pepsi

$1.29

- - - - - -

Unsure ' ' ' ' ' ' Certain

(Flip when estimate made and explanation complete.)

P_3: "How confident are you that you've made a good estimate?"

This is a grocery store ticket which has not yet been totaled. Estimate the total.

the KROGER co.

0.79	AGr
0.79	AGr
0.44	AGr
1.30	APr
0.34	APr
1.05	AGr
0.57	AMt
0.29	AGr
3.65	AGr
0.30	AGr
0.31	AGr
2.29	AGr
0.11	APr
0.34	APr
0.08	AGr

P_2: "Is there a different way you might have done this?"

- - - - - -

"You seem to have developed some very fine estimation skills, John. You've done an excellent job both on the test I gave you in class and in the questions I've asked you individually. Up to this point I have asked you to estimate in a variety of situations but haven't told you how accurate your estimates have been. In this last task I'd like you to estimate the answer to a few computation problems, then compare your estimate with the calculator result. Let's see how accurate your estimates are."

Examples: 42 + 23 + 7

17 x 20

- - - - - -

"I'd like you to use this calculator (HP-65), however, it works a little different than most calculators so I've programmed it to use algebraic logic. I'll have to reset it after each problem, though. Let's work a few examples so you'll know how to use it."

Press: 0, E.
42 + 23 + 7 Press: 42, enter, 23, enter, 7.
Reset
17 x 20 Press: 17, enter, 20, x.

"Are you ready? Let me reset it." Press: 10, E, f, STK.

"OK. Let's look at the first exercise. Write your estimate here, then compute the answer and write it here."

436 + 972 + 79

_ _ _ _ _

"I'll reset it." Press: 10, E.

42 963 ÷ 73

- - - - - -

Press: 10, E.

896 x 19

_ _ _ _ _ _ _

Press: 25, E, f, STK

896 + 501 + 789

- - - - - -

Before calculator is used:

P_4: "Do you think the actual answer will be above or below your estimate?"

Press: 25, E.

28 x 47

- - - - - -

P : "Tell me how you got your estimate."

P_3: "How confident are you that you've made a good estimate?"

Press: 50, E, f, STK

22 x 39

- - - - - -

Before the calculator is used:

P : Can you give me a better estimate?

Press: 50, E.

252 x 1.2

- - - - - -

Before the calculator is used:

P : "How sure are you of your estimate?"

At any appropriate time:

"Could you give me an upper bound for an estimate for this problem?"

"How sure are you of your estimate?"

- - - - - -

When student notices error:

"Why do you think that? Perhaps you made a keystroking error."

Let them verbalize error.

"When did you notice the calculator was making an error? Why didn't you tell me about it at that point?"

INTERVIEW PROBES

Alternate Route Probes

P_1: A student I interviewed last week estimated that answer to be _____. Is that a good estimate?

P_2: Do you see another way to get an estimate?

or

Is there another way you could do that?

Confidence Level Probe

P_3: How sure are you that you've made a good estimate?

Boundary Level Probes

P_4: Do you think the actual answer is above or below your estimate?

P_5: What would you say is the largest (smallest) number that would be a good estimate?

P_6: Which of these is the best estimate?

APPENDIX 5

Interview Summary Sheet

INTERVIEW SUMMARY GRADE ______ NAME ______________________ SCHOOL __________________ SEX _____ RACE ______

OPINION **COMMENTS**

Do you estimate? (where, how)

Is estimation part of mathematics?

Is it important?

OPINION (at close of interview)

What is estimation?

Have you used it?

Where learned?

Have you been taught how to estimate in school?

Practice?

Like it?

HINTS:

MPUTATION

EXERCISE	TIME	SCREENING TEST ANSWER	ESTIMATE	STRATEGY	PROBE	COMMENTS
1		______			P_4	
2		______			P_4	
3		______				
4		______			P_2	
5		______			P_3	Confidence ______

GENERAL COMMENTS

APPLICATION	TIME	SCREENING TEST ANSWER	ESTIMATE	STRATEGY	PROBE	COMMENTS
area					P_1 P_5	
30% of fans					P_4	
8483 x $.60					P_2	
total attendance					P_6	
divided proceeds						

EXERCISE	TIME	SCREENING TEST ANSWER	ESTIMATE	STRATEGY	PROBE	COMMENTS
ordering food					P_1	
dinner ticket					P_2	
most soda						
cheapest soda					P_3	Confidence ____
total for groceries					P_2	

CALCULATOR

how far?

probes used?

Comments by student

reaction to error

Level of confidence

unsure — very confident

Comments:

INTERVIEWER COMMENTS

Characterize strategies most often used: (rounding, front end, compensation . . .)

Specific unique strategies student uses:

Mental computer?

Variety of strategies?

Displays use of alternate routes to solution?

Consistency between screening and interview response:

General Comments:

Portion of tape to be transcribed: ______________________________

APPENDIX 6

Letter to Participating Teachers

Thanks for your help. This letter is intended to summarize the nature of our research project and highlight your individual committment.

Computational Estimation is recognized as a basic mathematics skill and is used more frequently than exact or precise computation. For example, suppose you have only $5 and want to purchase two cartons of milk at $1.79 each and three loaves of bread at $.59 each. Do you think you have enough money? This and most day-to-day mathematics problems rely heavily on computational estimation, yet virtually nothing is known about the processes (thinking strategies) used to solve them. The purpose of this research is to identify and describe successful and efficient computational estimation processes.

Although computational estimation is a term familiar to us as teachers, it is both hard to describe and develop within our students. It can be defined as the interaction and/or combination of mental computation, number concepts, technical arithmetic skills including rounding, place value, and less straightforward processes such as mental compensation that rapidly and consistently result in answers that are reasonably close to a correctly computed result. This process is done internally without the external use of a calculating or recording tool.

In order to identify students who have developed efficient computational estimation strategies, a screening device has been developed to allow for group administration and to carefully control timing on each exercise. This screening instrument presents a number of computation exercises, some within an applied context. Each exercise is contained on a slide which is projected on a screen in the classroom. Students will be given a limited amount of time to estimate the answer to each exercise. The administration of this screening device will require from 20-30 minutes.

Prior to giving the Computational Estimation test to your students, we would like to visit with you a few minutes (probably 10 or so) to answer any questions related to this study. This also provides an opportunity to get your reaction to some characteristics which we think good estimators will possess. The test will be given to approximately 400 children in this grade level and when all of the tests have been scored, we will report the results of your class to you, if you wish. Since we are trying to find the strategies used by good estimators, the top 5% or so of the students will become candidates for the individual interviews. If any of these top students are in your class, we will be checking with you to learn more about them and probably try to schedule two interviews, about 30-40 minutes each.

It is anticipated that the identification and characterization of successful and efficient computational estimation strategies will contribute to the formulation of a general cognitive framework. Furthermore, it is hoped that this framework will guide future curricular and instructional development in school mathematics and adult basic education.

APPENDIX 7

Teacher Recommendation Form

Teacher Recommendation Form

Name ____________________

School __________________

At this stage of our research, we think good estimators will have some or all of the following characteristics:

1. Quick with paper and pencil computations.
   Among the first to respond to oral questions and/or hand in their test papers.
2. Accurate with arithmetic computations.
   Check computations and strive for a high degree of accuracy.
3. Unafraid to be wrong.
   Risk contributing probable solutions to problems, easily cope with being wrong, and continue to probe for the solution.
4. Mathematical confidence.
   Possess good computational skills and realize potential to compute.
5. Demonstrated performance.
   Demonstrate adequate estimation skills and use them regularly.
6. Mathematical judgement.
   Judge a problem situation and determine when an estimate is appropriate and when an exact solution is needed.
7. Reasonableness of answers.
   Sense when an answer is not in the ballpark. Able to reject far out answers and seek more reasonable results.
8. Divergent thinking strategies.
   Have a knowledge of a variety of strategies and a tendency to search for alternate routes to a solution for a given problem.

Please list students from your class who you believe are good computational estimators. (If more room is needed, use back side.)

APPENDIX 8

Transcriptions of Calculator Portion of Three Interviews

(7th grader)--Verbalization of Error at Exercise 3

Exercise 1: 436 + 972 + 79

Student Estimate: 1472   Calculator Response: 1627

Student: 1472

Interviewer: Now use the calculator to find out the exact answer.

S: Uhh...I did it bad.

I: So the exact answer is 1627?

S: That's pretty bad...is it 1627?

I: Is that what the calculator said?

S: Yes, it doesn't look right, though.

I: What do you mean?

S: Looks like less than 1627 to me...79 + 972 is about 1050 + 436 looks like 1500 at the most.

I: It says 1627, though?

S: Well maybe it's wrong, I don't know.

I: Maybe you punched the buttons wrong.

S: I could have. (Repunches problem.)

S: It doesn't look right, but if that's what the calculator says, then it's probably right. It still doesn't look right.

I: Let's look at the next one.

Exercise 2: 42 962 ÷ 73

Student Estimate: 600 Calculator Response: 638.52

-----No comment-----

Exercise 3: 896 x 19

Student Estimate: 17 000 Calculator Response: 18 724

S: I must have made a mistake somewhere. It still doesn't look right.

I: What do you mean?

S: 20 x 900 is 18 000. I don't see how it could be any more than 18 000.

I: But the calculator gave 18 724.

S: You're doing something with the calculator, I think.

I: What do you mean?

S: Like trying to make it wrong. So I think it's wrong.

I: You think the calculator is wrong?

S: Yes, I do.

I: Are you sure?

S: Yes, I think so.

I: You think so?

S: Yes, let's just say positive.

(9th grader)--Verbalization of Error in Discussion Following Exercise 7

Exercise 1: 436 + 972 + 79

Student Estimate: 1480 Calculator Response: 1627

S: So I was about 140 off.

Exercise 2: 42 962 ÷ 73

Student Estimate: 613 Calculator Response: 638.52

S: This time I was off by 25.

Exercise 3: 896 x 19

Student Estimate: 17 920 Calculator Response: 18 724

S: I did that by 896 x 20. I was off by about 800. I must have made a mistake. That would be right because I rounded it up one number, that would have been another 800.

Exercise 4: 896 + 501 + 789

Student Estimate: 2186 Calculator Response: 2726

S: I was off about 600.

Exercise 5: 28 x 47

Student Estimate: 1410 Calculator Response: 1636

I: Before you use the calculator, tell me how you estimated this one.

S: I rounded to 30 x 47. If I wanted more accuracy I would subtract 50. So I could have 1360.

I: How confident are you of your estimate?

(Pointed to the middle of the scale.)

S: 1636! Off about 300.

I: What do you think?

S: I can't figure out why my estimate is wrong unless my multiplication is wrong.

Exercise 6: 22 x 39

Student Estimate: 858 Calculator Response: 1278

I: How did you do that?

S: I rounded to 20 x 39. That gives 780 then add 39 twice or 78. So it's 858.

(uses calculator)

S: 1278. I'm getting confused.

I: What do you mean?

S: I know what I keep forgetting...I can't figure out why there is a 400 difference there. If I do it 22 x 40 = 880 then subtract off 39 I get 841.

I: What do you think?

S: I have two answers here around 850 but their correct answer is around 1200 so somewhere I made a mistake.

I: Do you think you multiplied wrong somewhere?

S: That's why I checked it by doing it another way.

I: So what do you think?

S: Hmm...I'm not sure if it's my multiplication.

I: What else could it be?

S: I don't know, maybe I'm not doing the problem right.

Exercise 7: 252 x 1.2

Student Estimate: 312.4 Calculator Response: 452.4

S: I multiplied 252 x 1 then 252 x .2 then added them together.

I: So should that be pretty close?

S: The way I'm going, I wouldn't say anything right now.

(uses calculator)

I: Do you think that's pretty close?

S: My answer is about three-fourths of it.

I: So you're satisfied with it?

S: Yes, pretty much. More than the last one.

I: As we look back, it looks like all of the exact answers are higher than your estimates.

S: Yes, I usually underestimate except with money.

I: Let's look at 28 x 47, how did you do that one?

S: 30 x 47 = 1410 then I subtract...I mean add 47 twice so I'd get 1504.

I: What do you think about your estimates.

S: They're pretty...OK, I guess, except for 22 x 39.

I: Why?

S: I don't understand why my numbers are so far off. Can I round down the numbers?

I: Yes.

S: I don't know what to think.

I: What could be wrong?

S: I don't know.

I: Is that the only one that bothers you?

S: No, the first one too. The way I rounded, it shouldn't be 200 off. The other addition one I'm not satisfied with either. The others are close enough. If you round this one (fourth) to 900 + 500 + 800 you get 2200.

I: So what could be wrong?

S: Something's cooky with the calculator on these (first, fourth, and sixth).

(12th grader)--Verbalization of Error at Exercise 3

Exercise 1: 436 + 972 + 79

Student Estimate: 1490 Calculator Response: 1627

I: So the exact answer is 1627?

S: How did I manage to miss by over 100? Wait a minute...that should have been 1590.

I: So you made a small error?

S: Trouble is, when you make a small error on the higher digits it multiplies it by 10 for every space over.

I: Let's try the next one.

Exercise 2: 42 962 ÷ 73

Student Estimate: 580 Calculator Response: 638.52

S: Most of what I get are ballpark figures.

I: So the exact answer is what? 638.52.

(student hesitates)

S: Yeah...it's just a ballpark figure. I can hit it within about 10 percent.

I: Let's try another one.

Exercise 3: 896 x 19

Student Estimate: Calculator Response: 18 724

S: That's obviously not the correct answer.

I: What do you think is the matter?

S: Something is fishy with your calculator.

I: Maybe the multiplication key is messed up.

S: See that's one thing I often use, I can tell when my calculator batteries are dead because it starts giving me fishy answers.

I: Do you think maybe it's just the multiplication key?

S: You might have them switched around on me.

I: I mean, it was alright on the first two problems.

S: Well, it occured to me that it smelled slightly here (division exercise).

I: What do you mean?

S: The result here was larger than I expected. Because it would seem to me that 73 adds a greater ratio over 70 than does 4300 over 4200 so that the answer should have been less than 600.

I: What about the first problem?

S: It looks like it's built up a bit, too.

I: What do you think?

S: I have a sneaking suspicion that your calculator is multiplying it (the answer) by a certain small constant.